Balanced Trees

Part Two

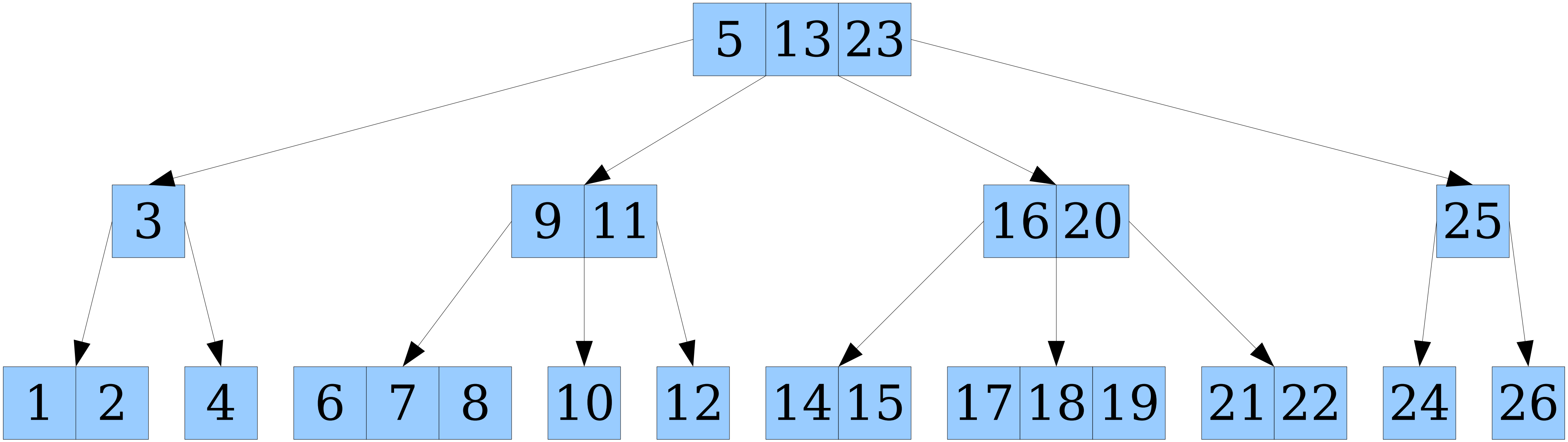
Outline for Today

- ***Red/Black Trees***
  - Using our isometry!
- ***Tree Rotations***
  - A key primitive in restructuring trees.
- ***Augmented Binary Search Trees***
  - Leveraging red/black trees.

Recap from Last Time

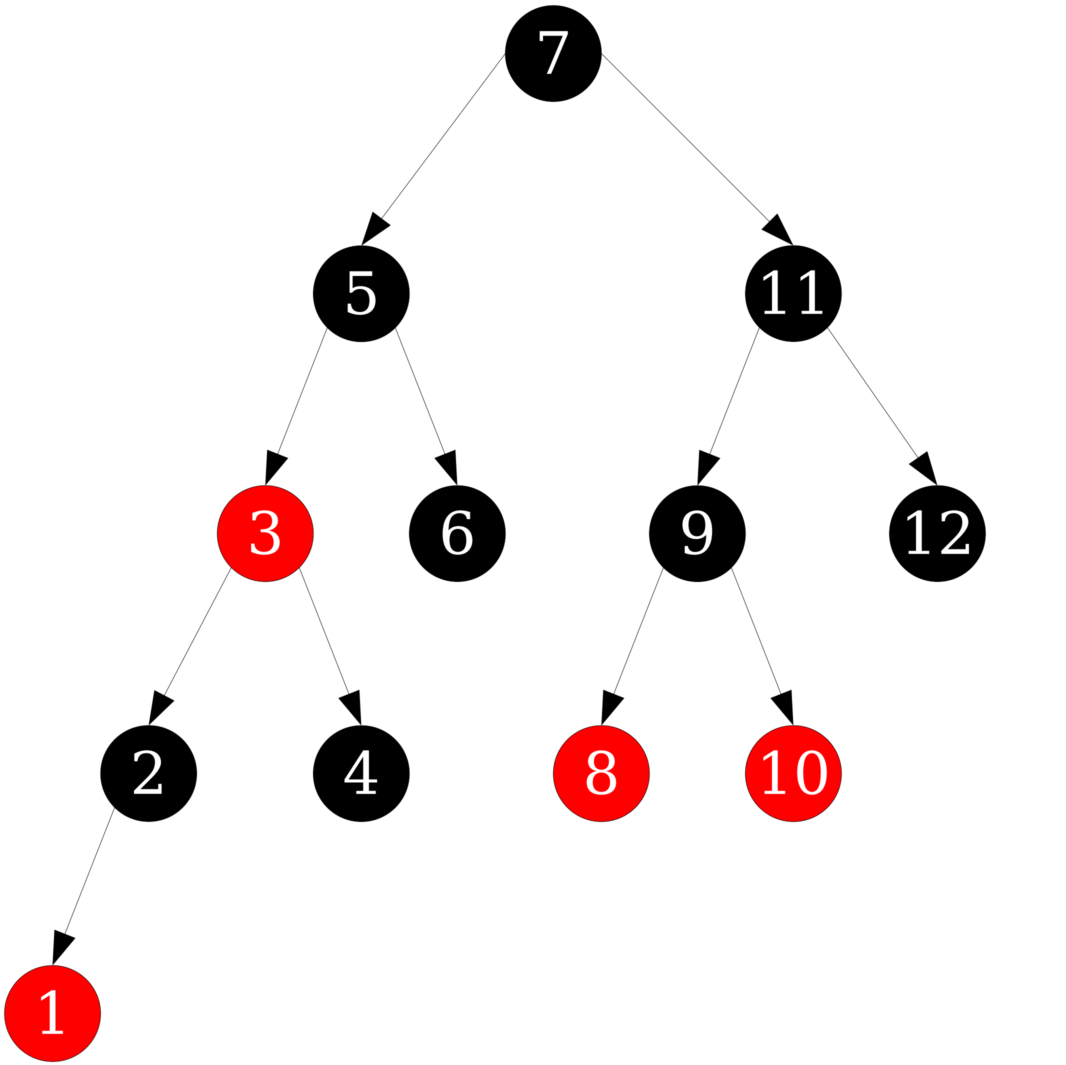
2-3-4 Trees

- A ***2-3-4 tree*** is a multiway search tree where
  - every node has 1, 2, or 3 keys,
  - any non-leaf node with k keys has exactly $k+1$ children, and
  - all leaves are at the same depth.
- To insert a key, place it in a leaf. If out of space, split the leaf and kick the median key one level higher, repeating this process.



Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



Red/Black Trees

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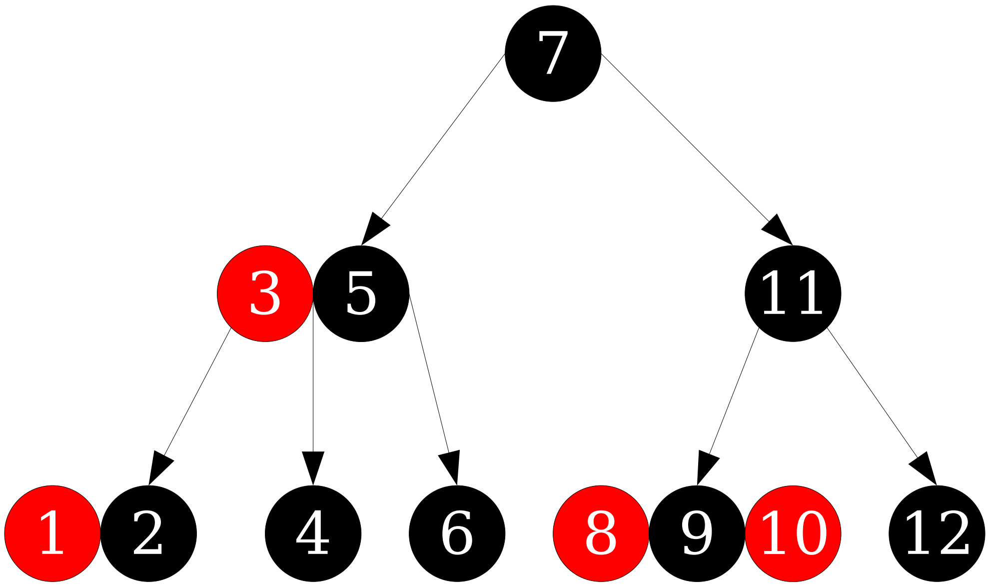
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.
- After we hoist red nodes into their parents:
  - Each “meta node” has 1, 2, or 3 keys in it. (No red node has a red child.)
  - Each “meta node” is either a leaf or has one more child than key. (Root-null path property.)
  - Each “meta leaf” is at the same depth. (Root-null path property.)



This is a 2-3-4 tree!

New Stuff!

Data Structure Isometries

- Red/black trees are an ***isometry*** of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- That gives us some really easy theorems basically for free.
- ***Theorem:*** The maximum height of a red/black tree with n nodes is O(log n).

Data Structure Isometries

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- That gives us some really easy theorems basically for free.
- ***Theorem:*** The maximum height of a red/black tree with n nodes is O(log n).

Explain why, using the isometry.

Answer at
https://cs166.stanford.edu/pollev

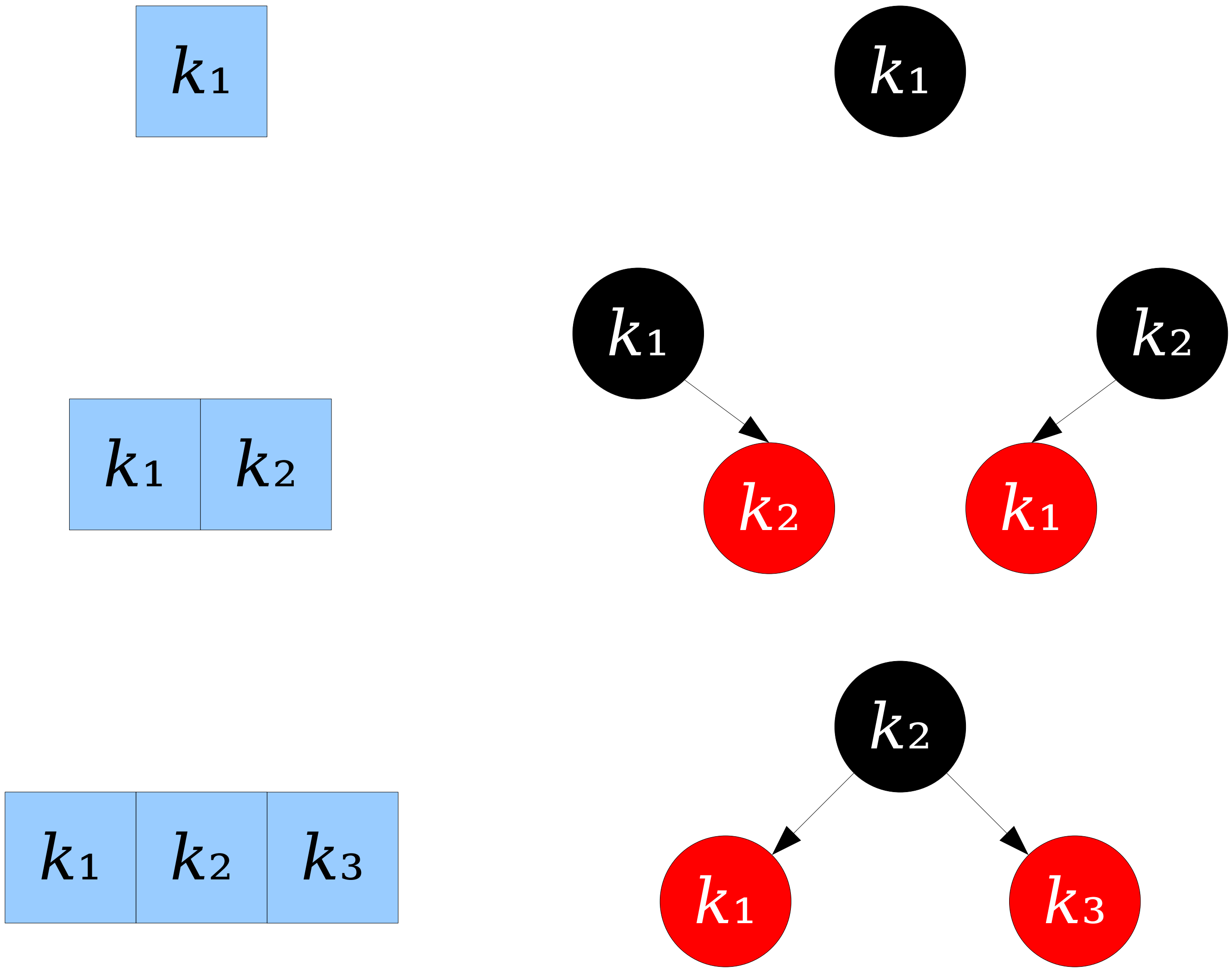
Data Structure Isometries

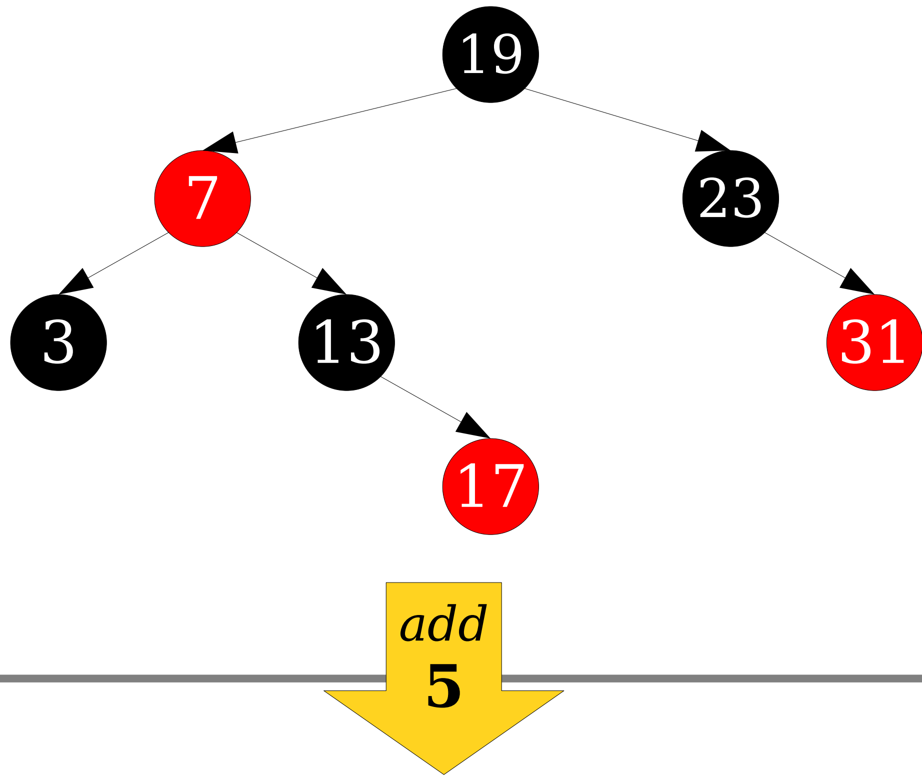
- Red/black trees are an ***isometry*** of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- That gives us some really easy theorems basically for free.
- ***Theorem:*** The maximum height of a red/black tree with *n* nodes is O(log *n*).
- ***Proof idea:*** Pulling red nodes into their parents forms a 2-3-4 tree with *n* keys, which has height O(log *n*). Undoing this at most doubles the height of the tree. ■-*ish*

Exploring the Isometry

- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - ***2-nodes*** have one key (two children).
  - ***3-nodes*** have two keys (three children).
  - ***4-nodes*** have three keys (four children).
- How might these nodes be represented?

Exploring the Isometry

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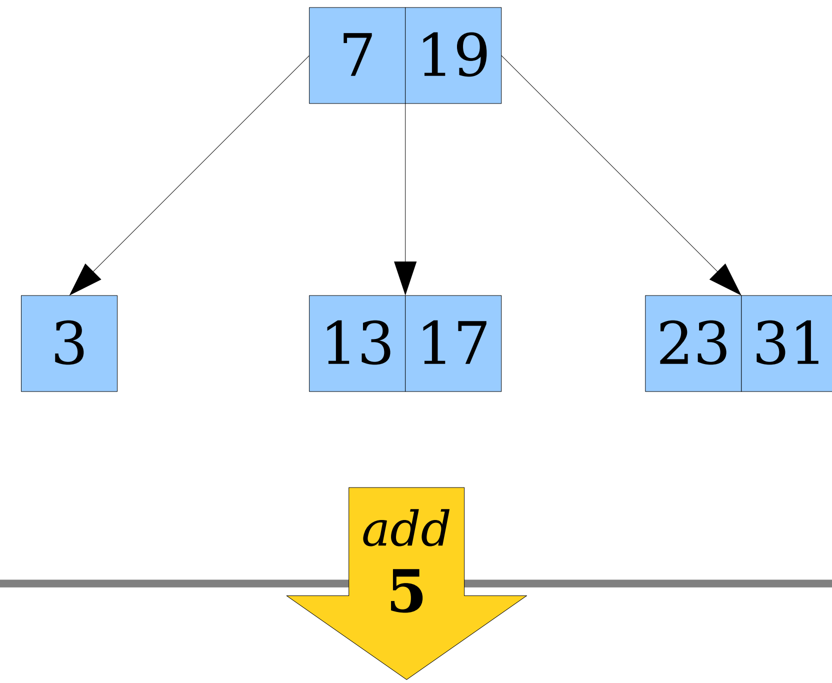
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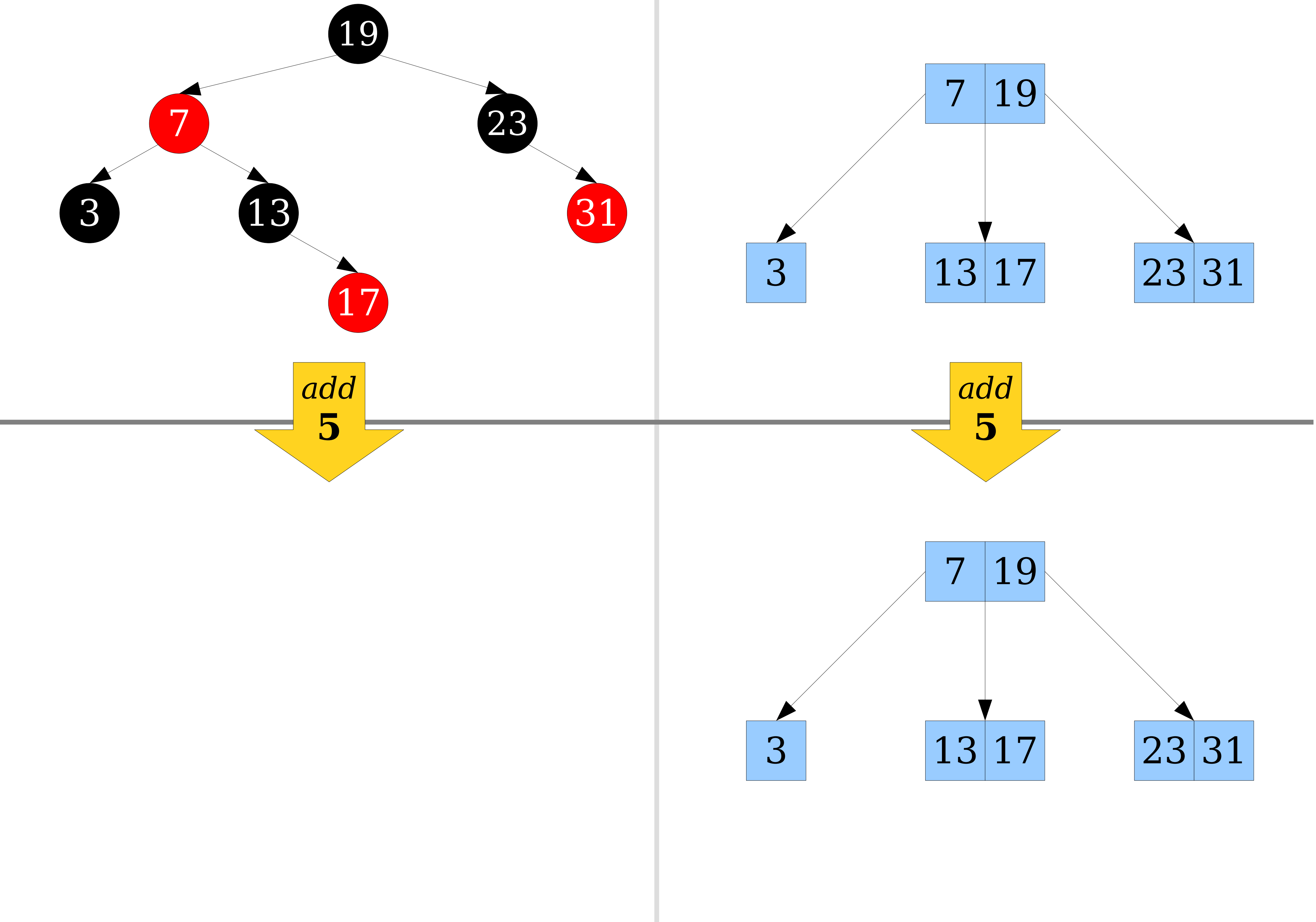
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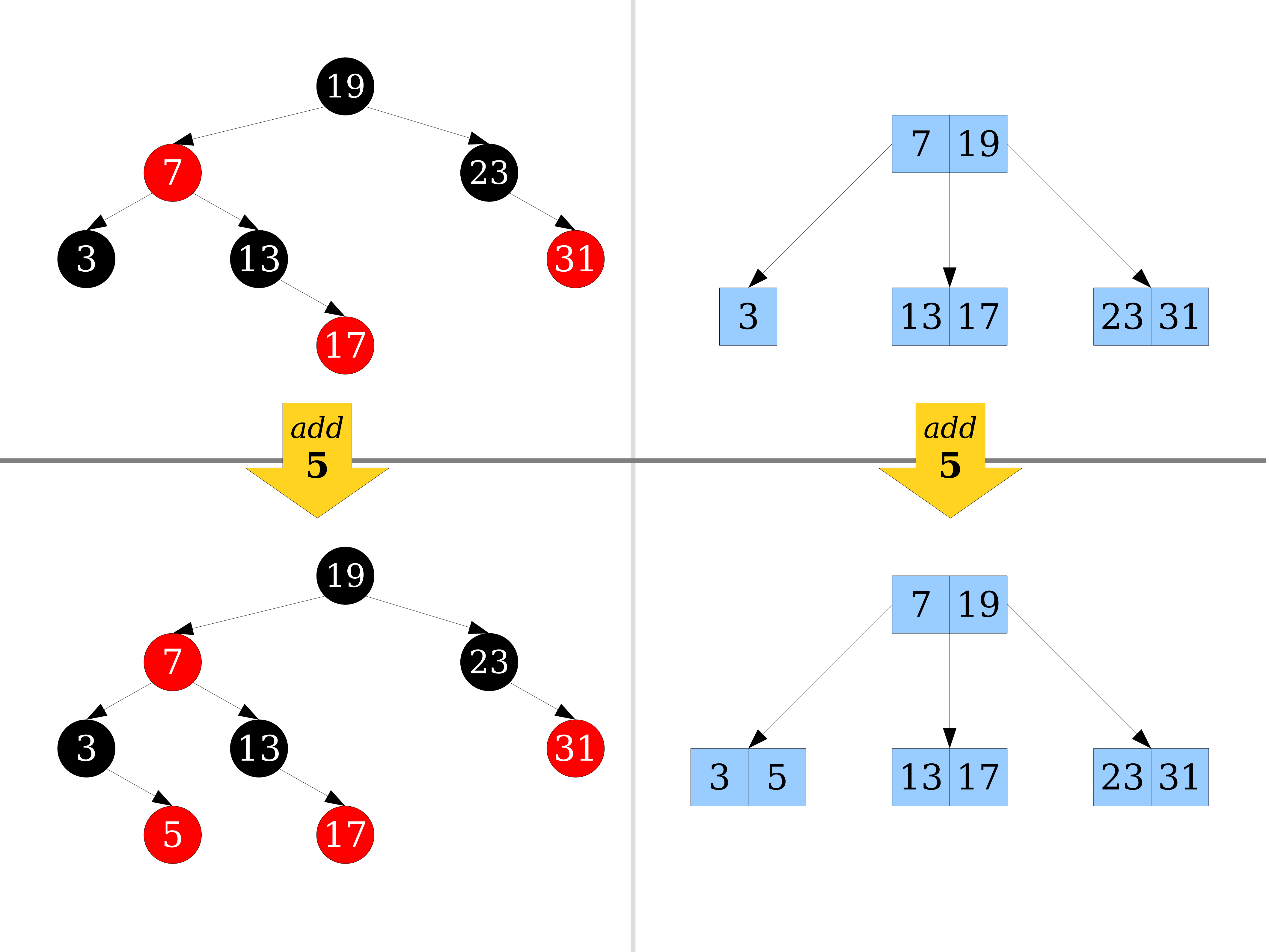
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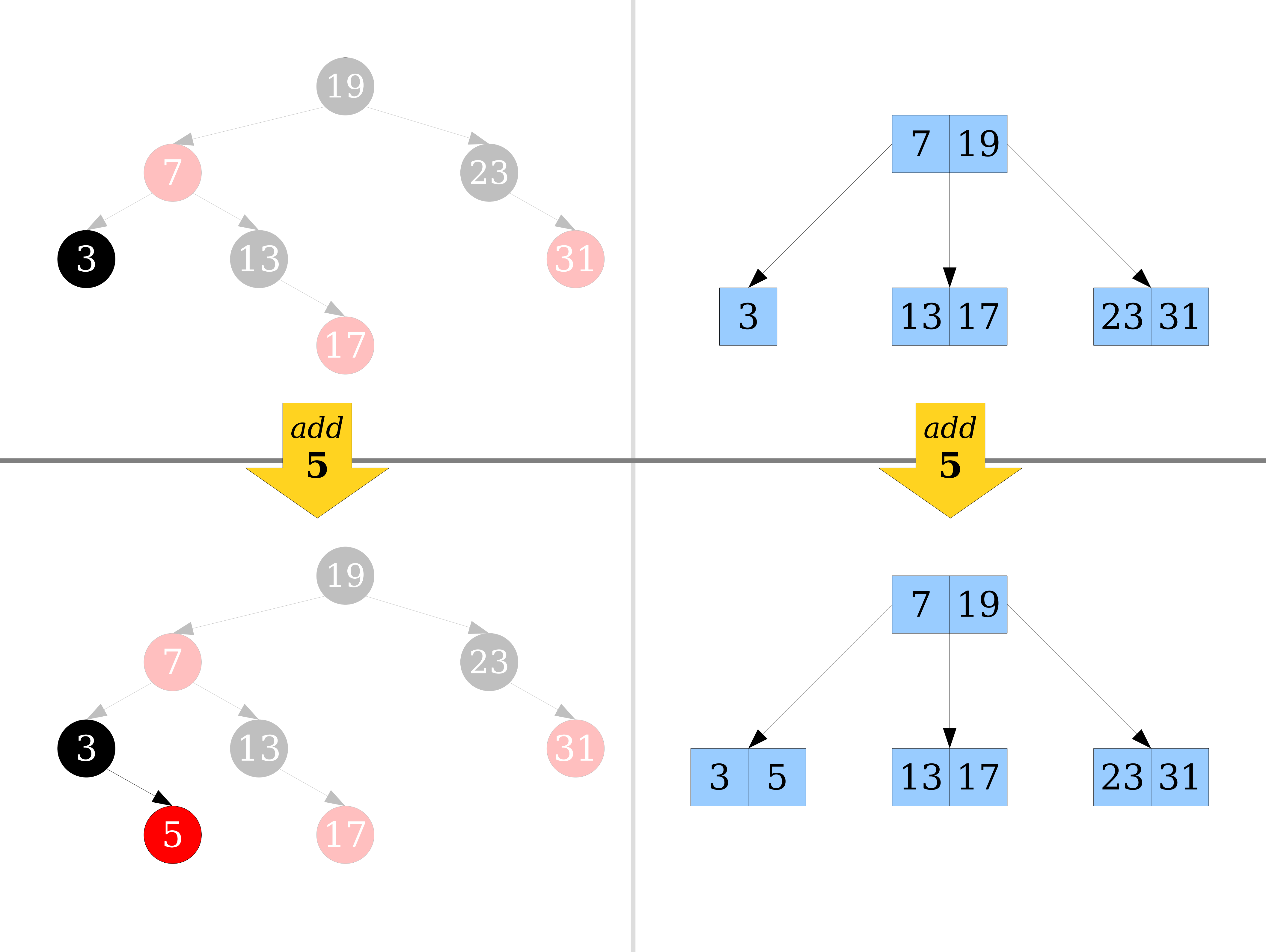
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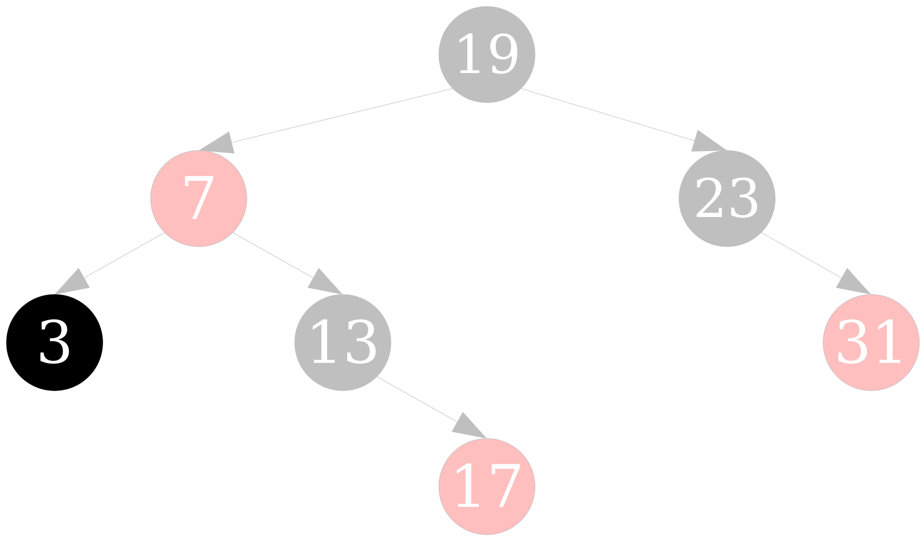

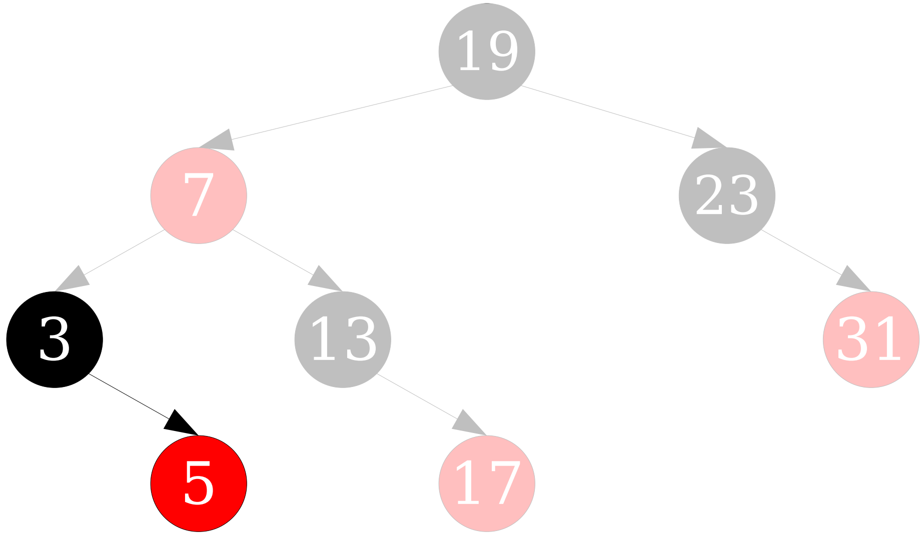

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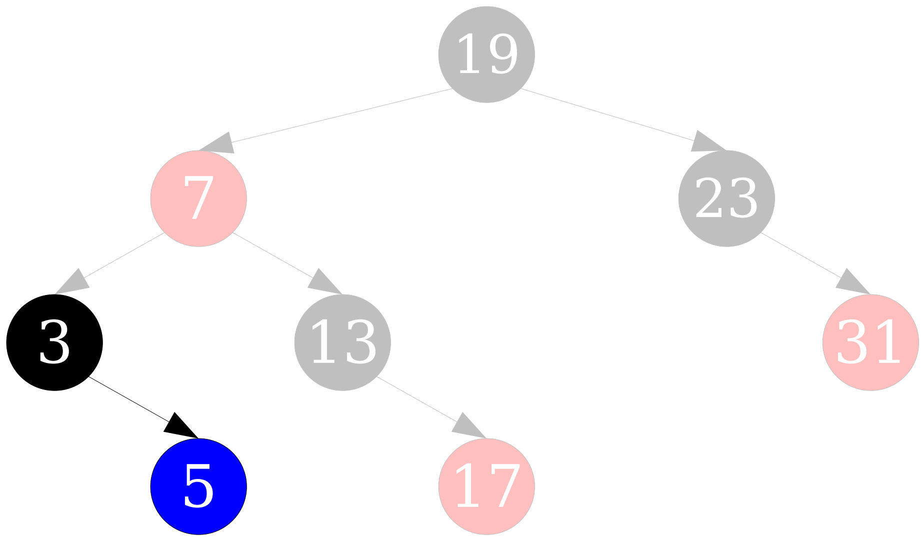

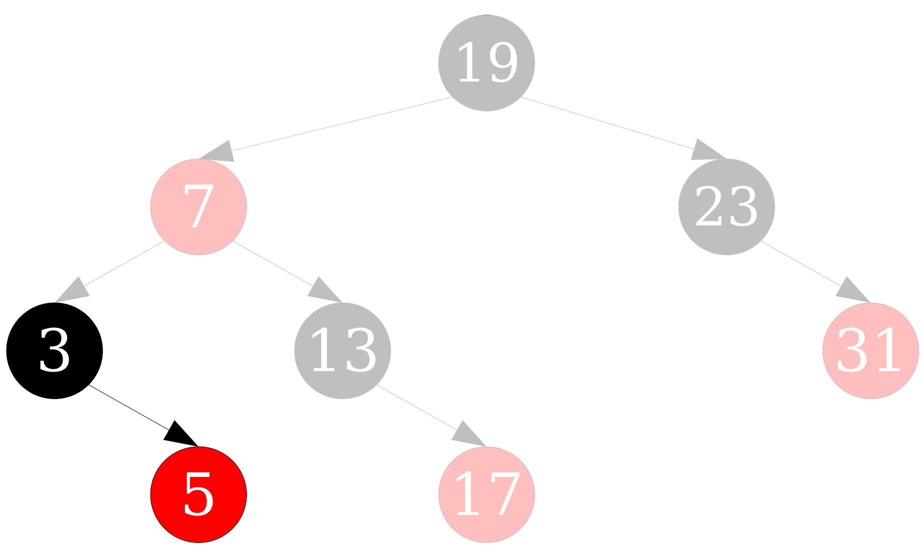

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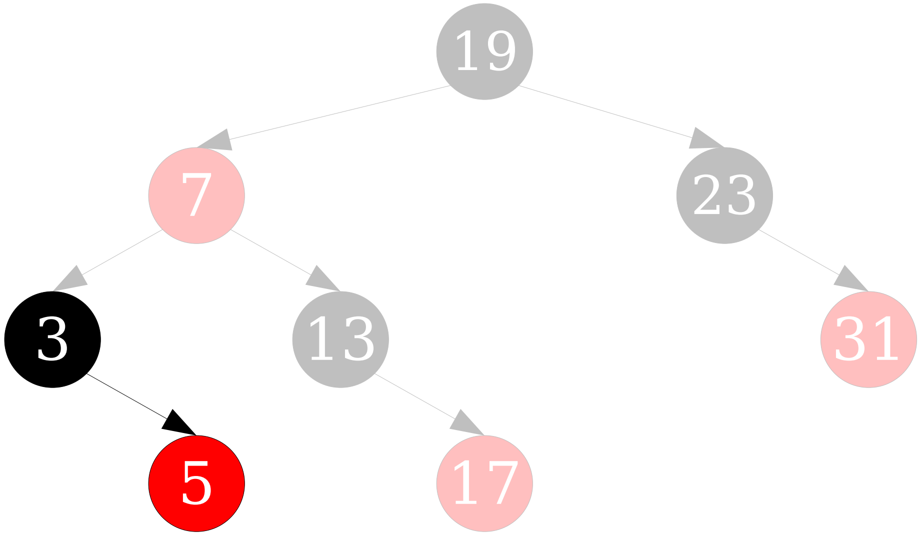

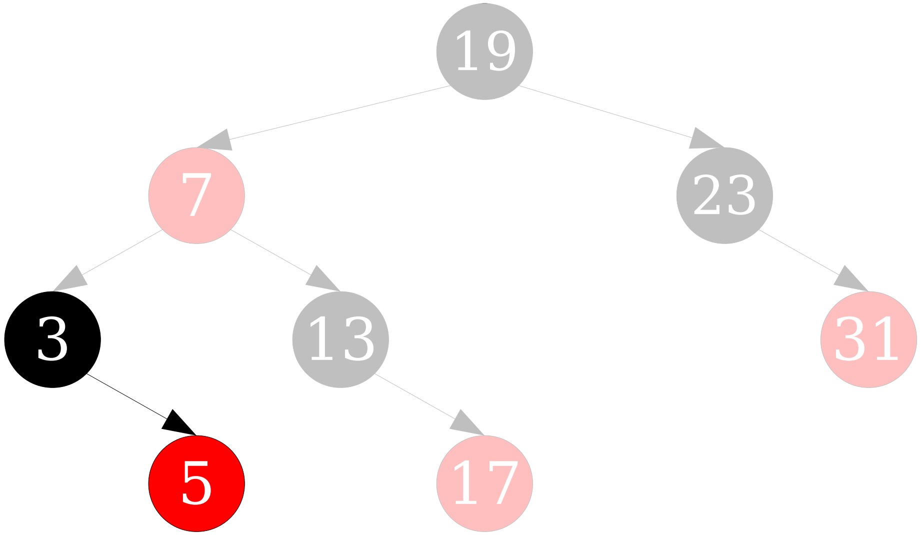

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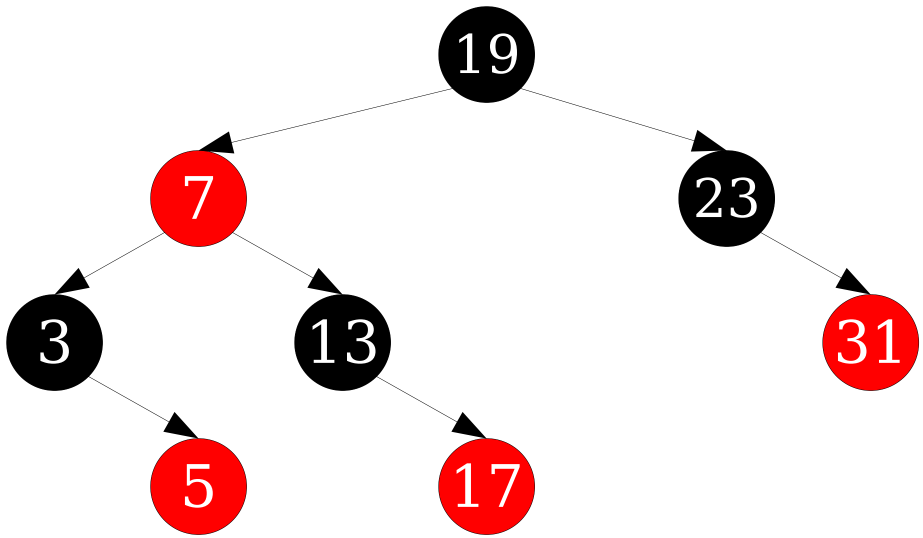

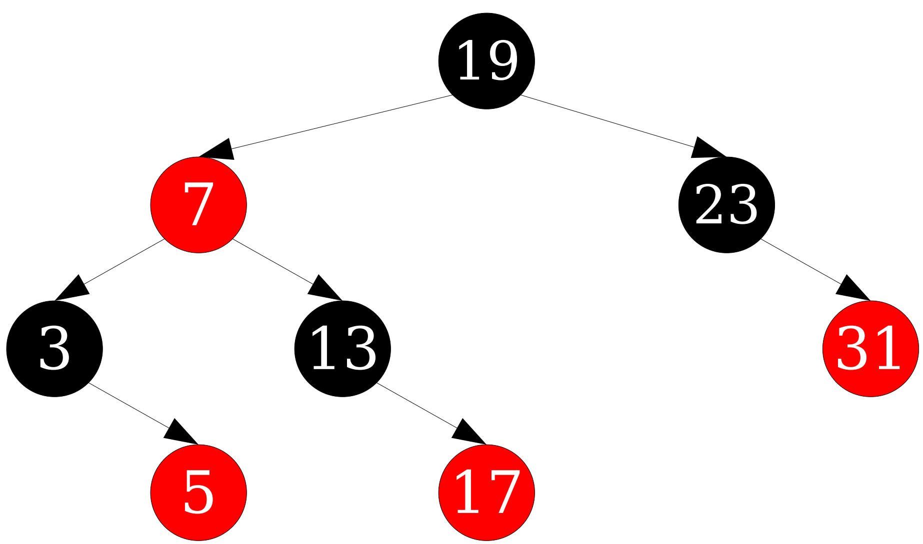

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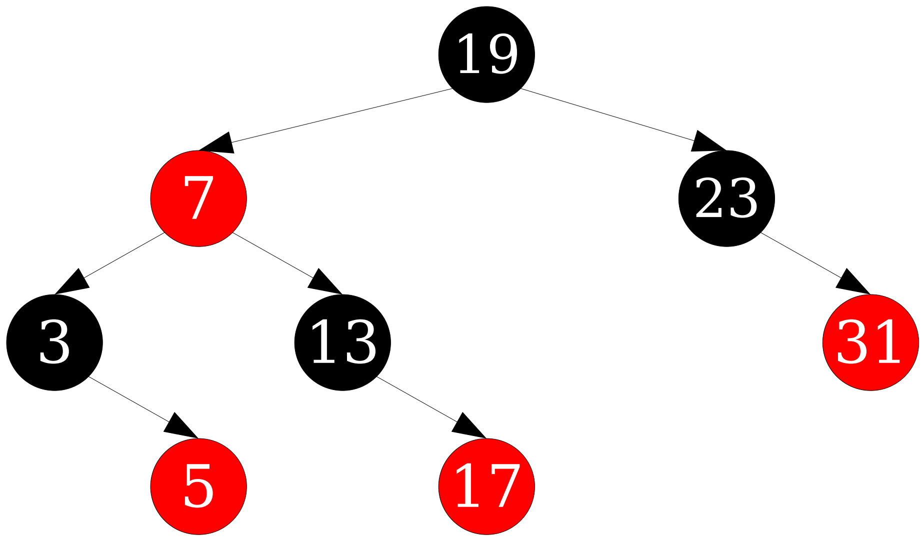
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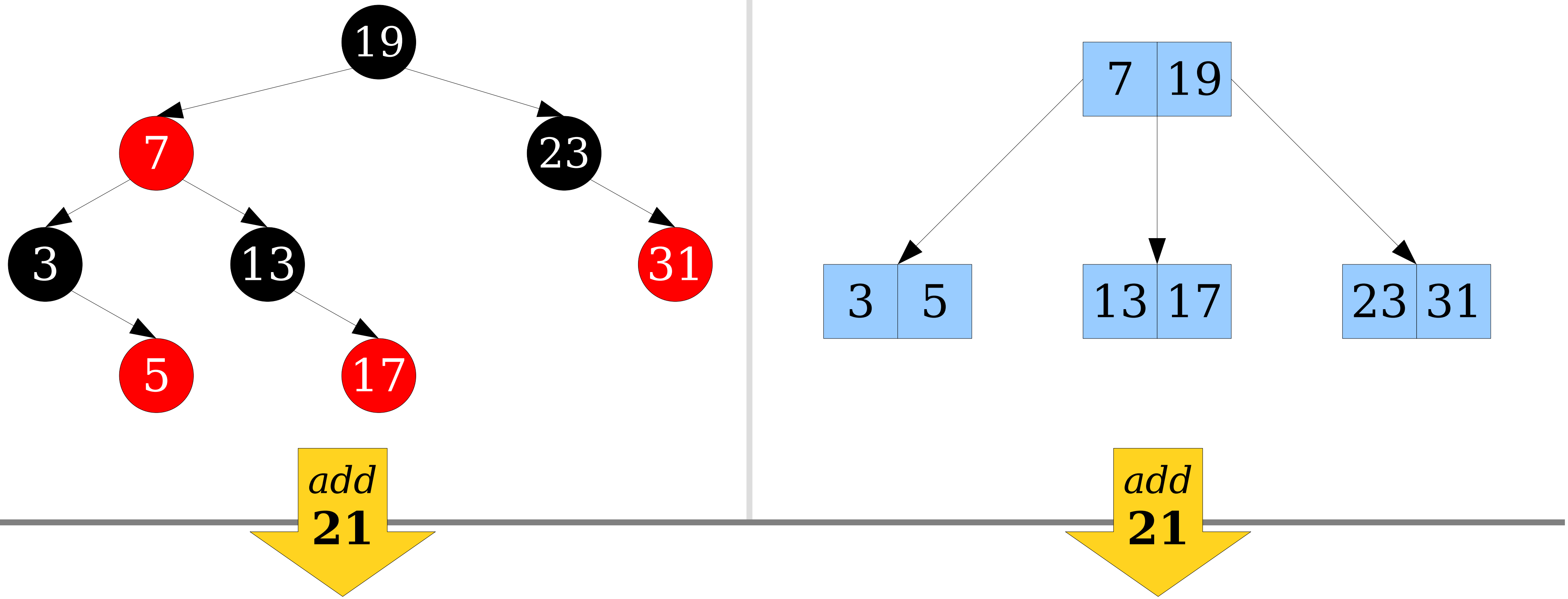
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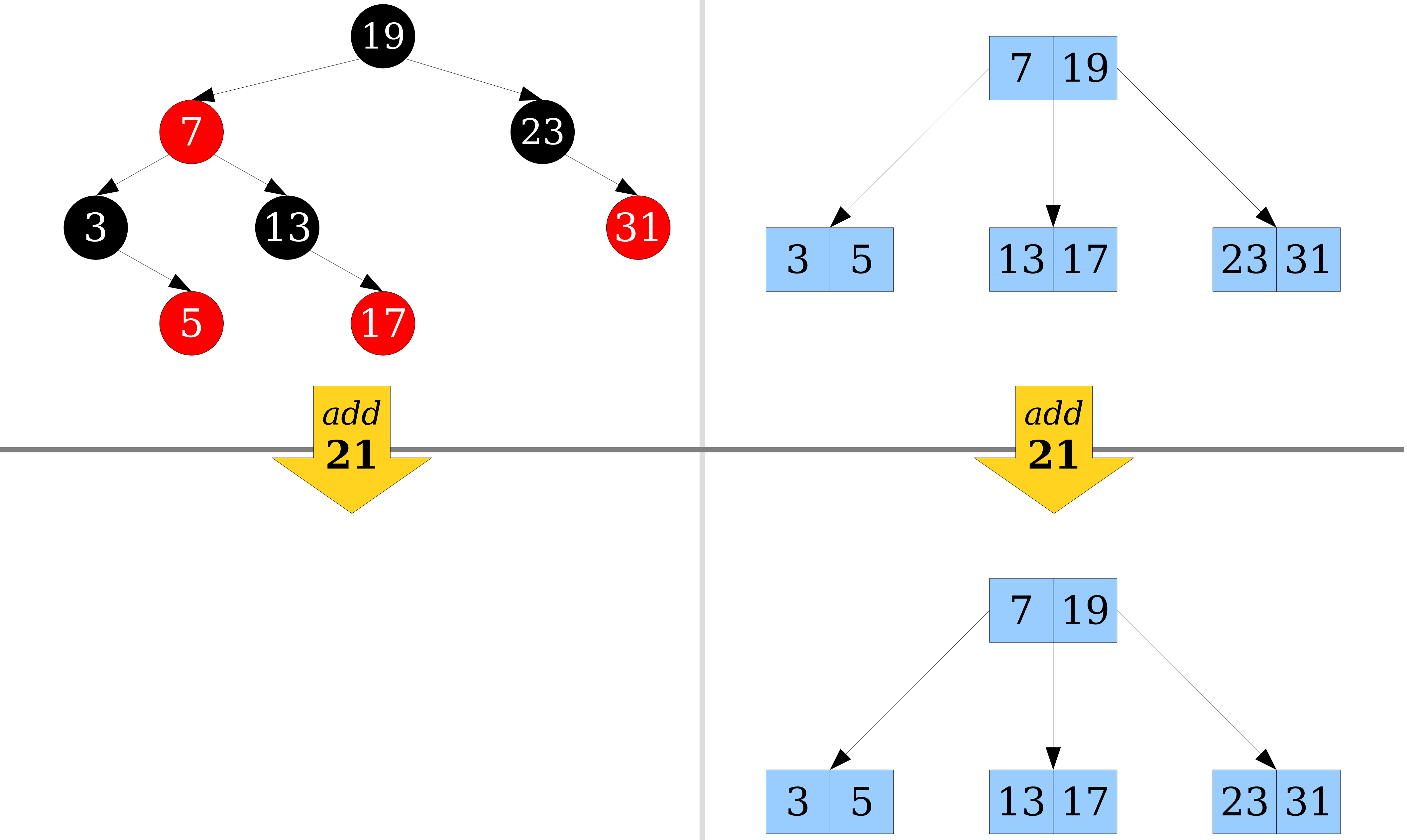
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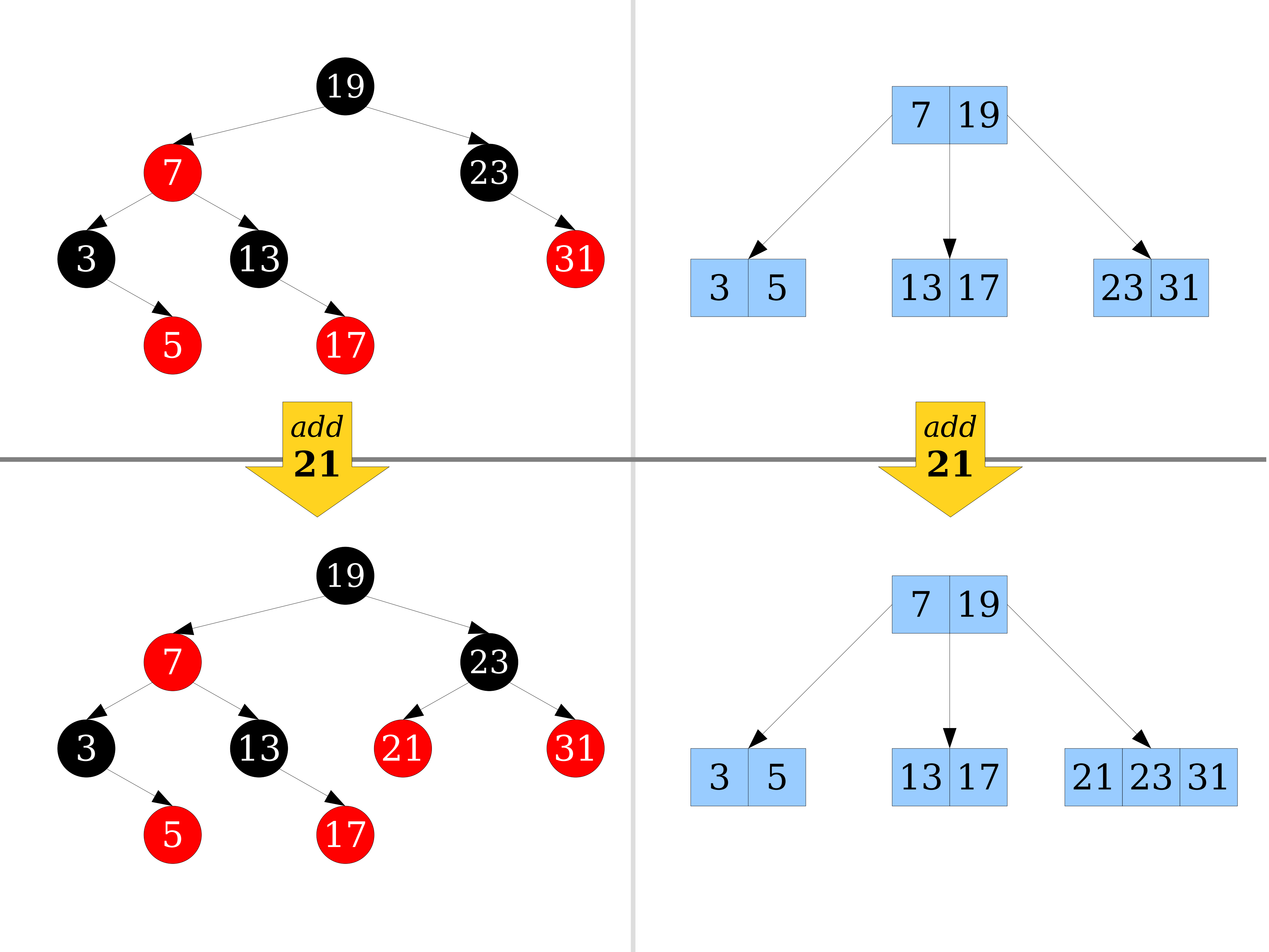
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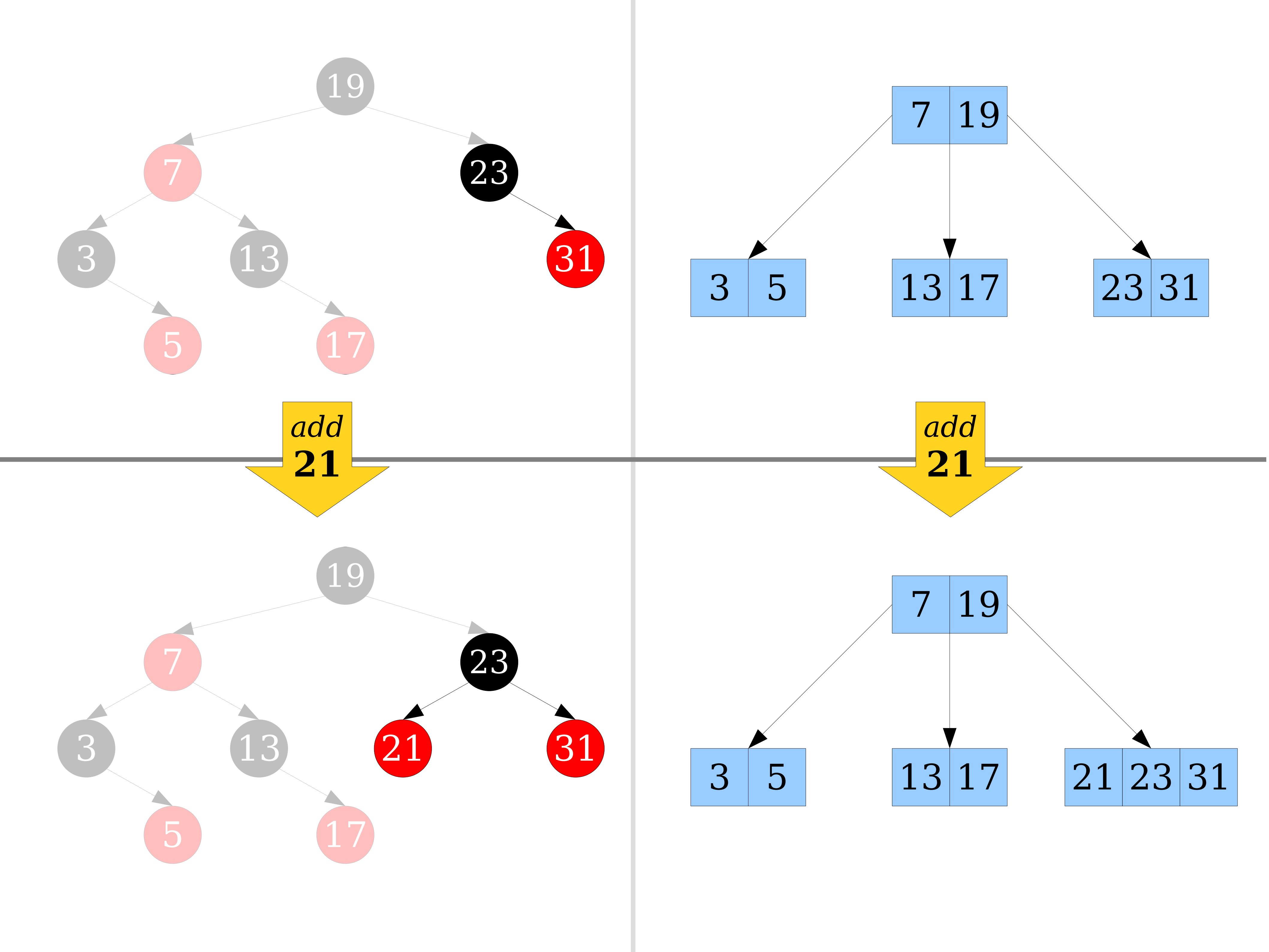
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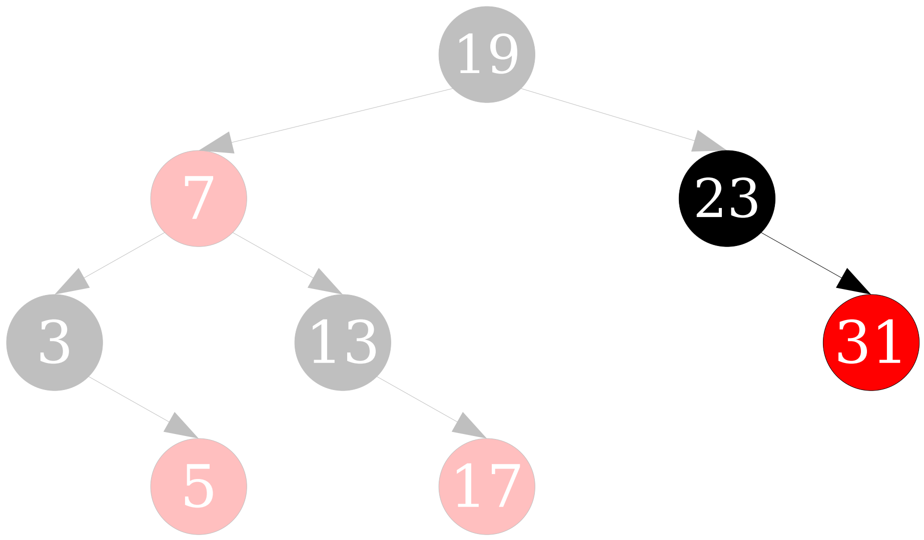

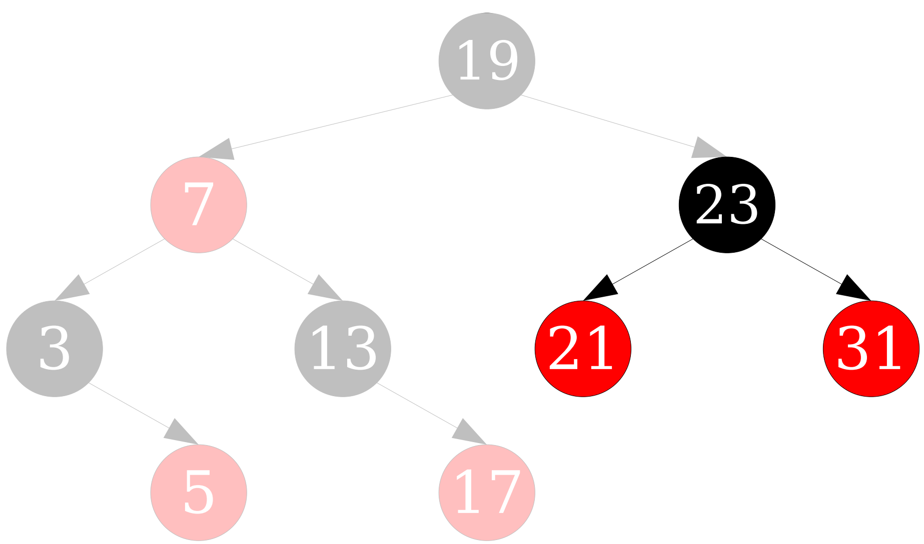

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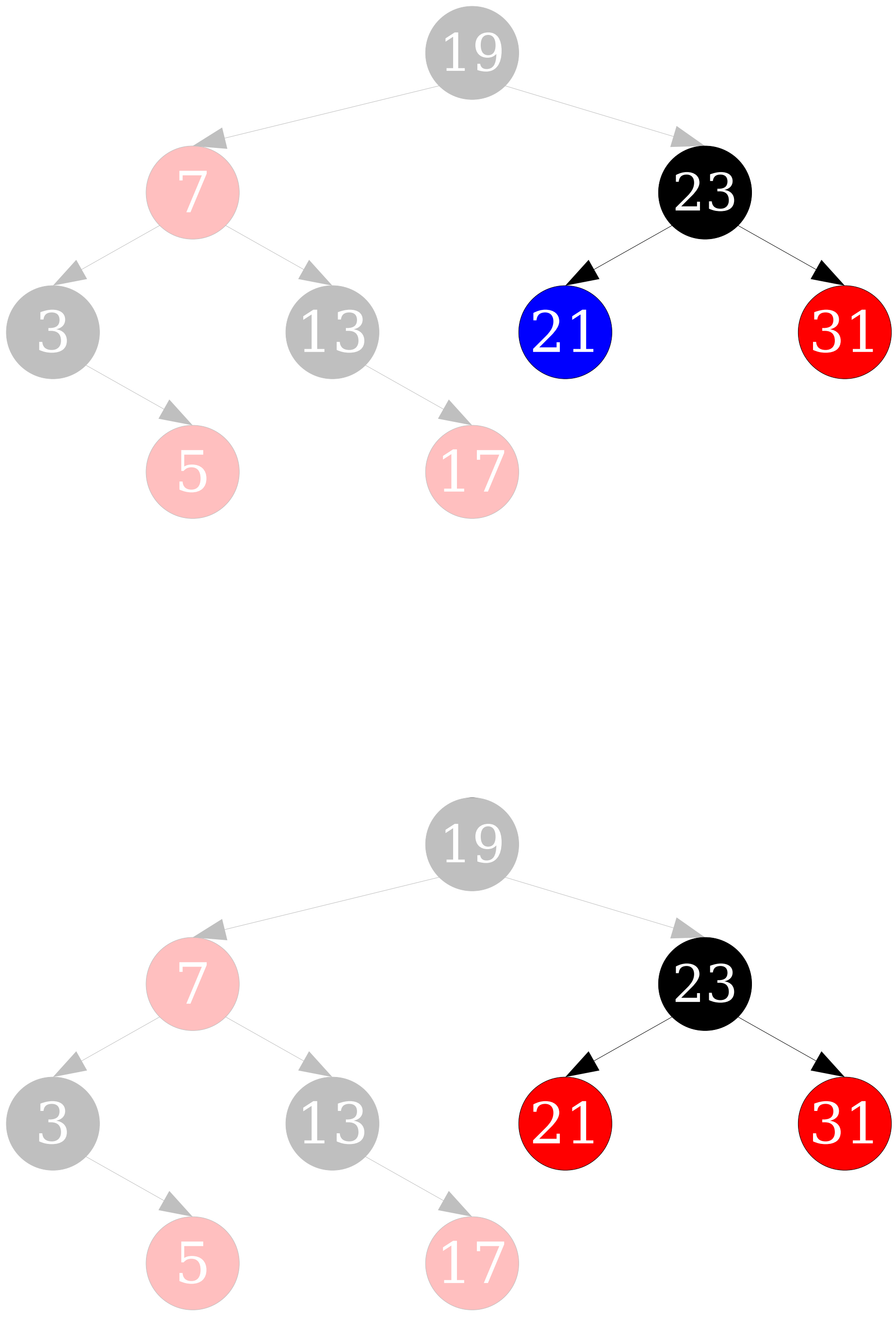

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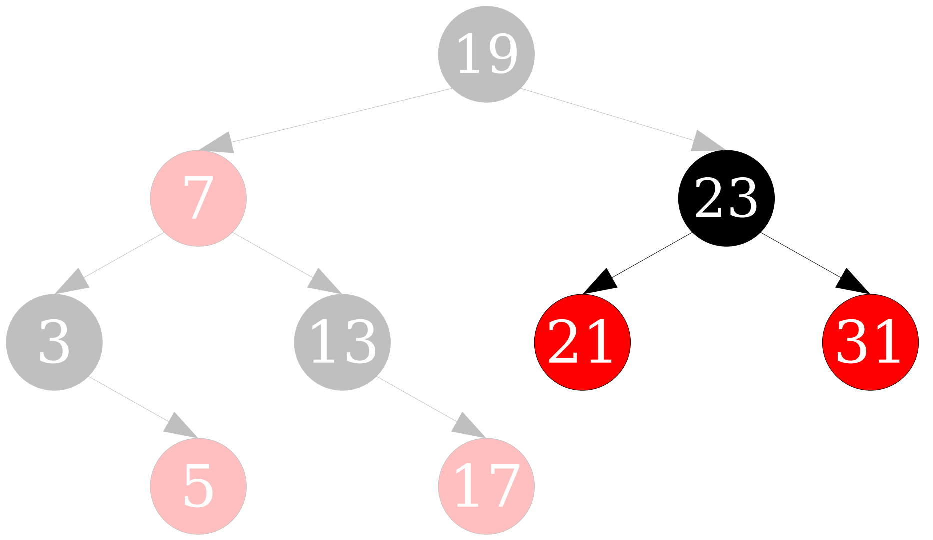

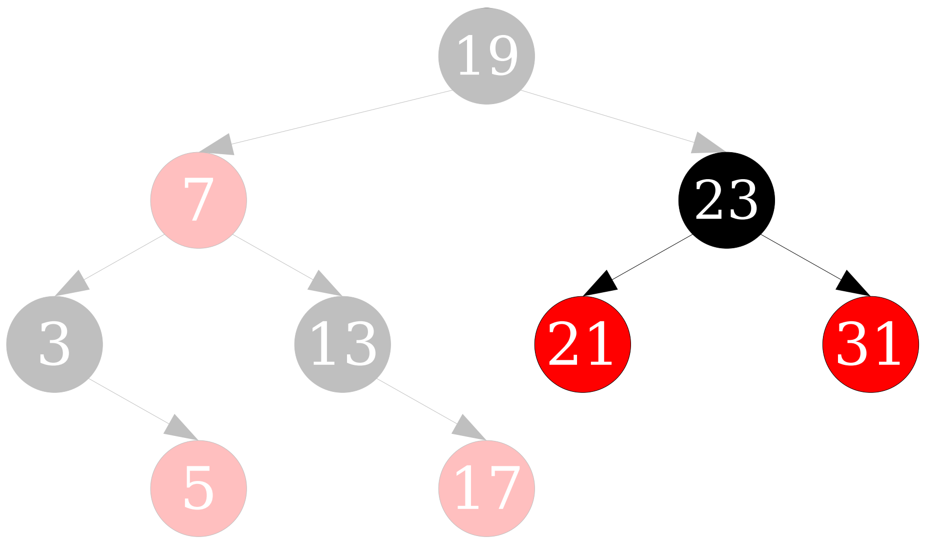

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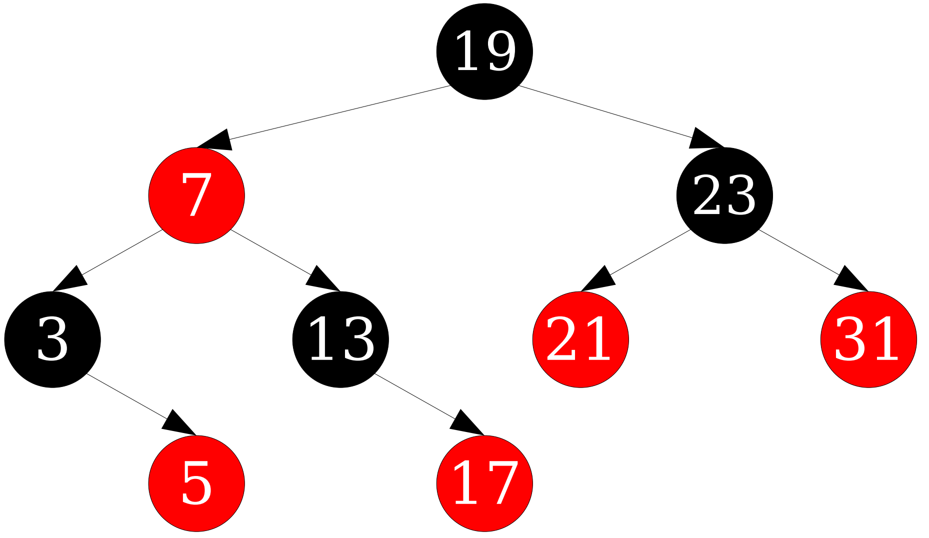
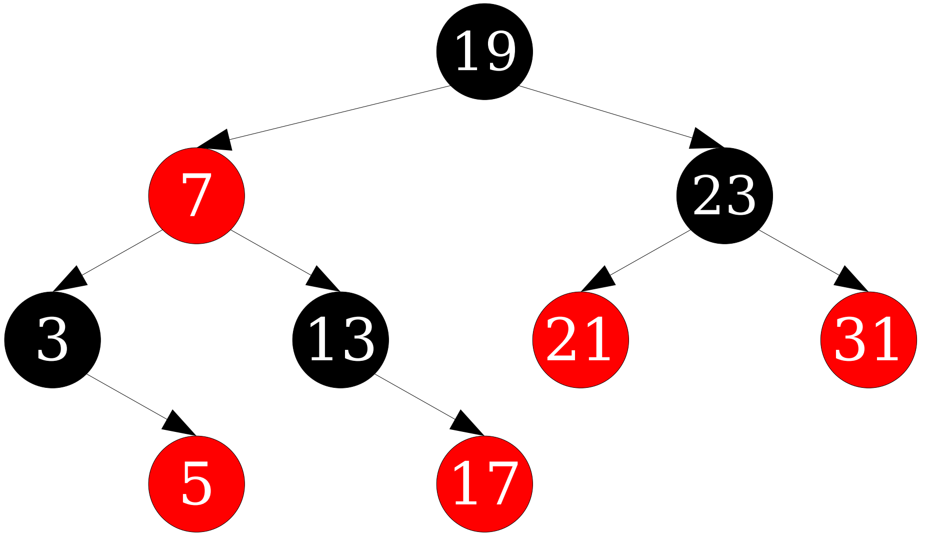
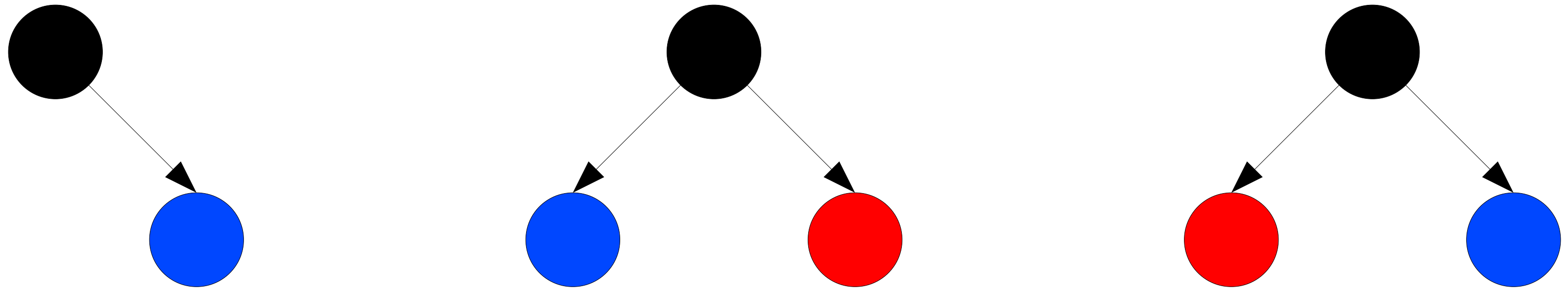


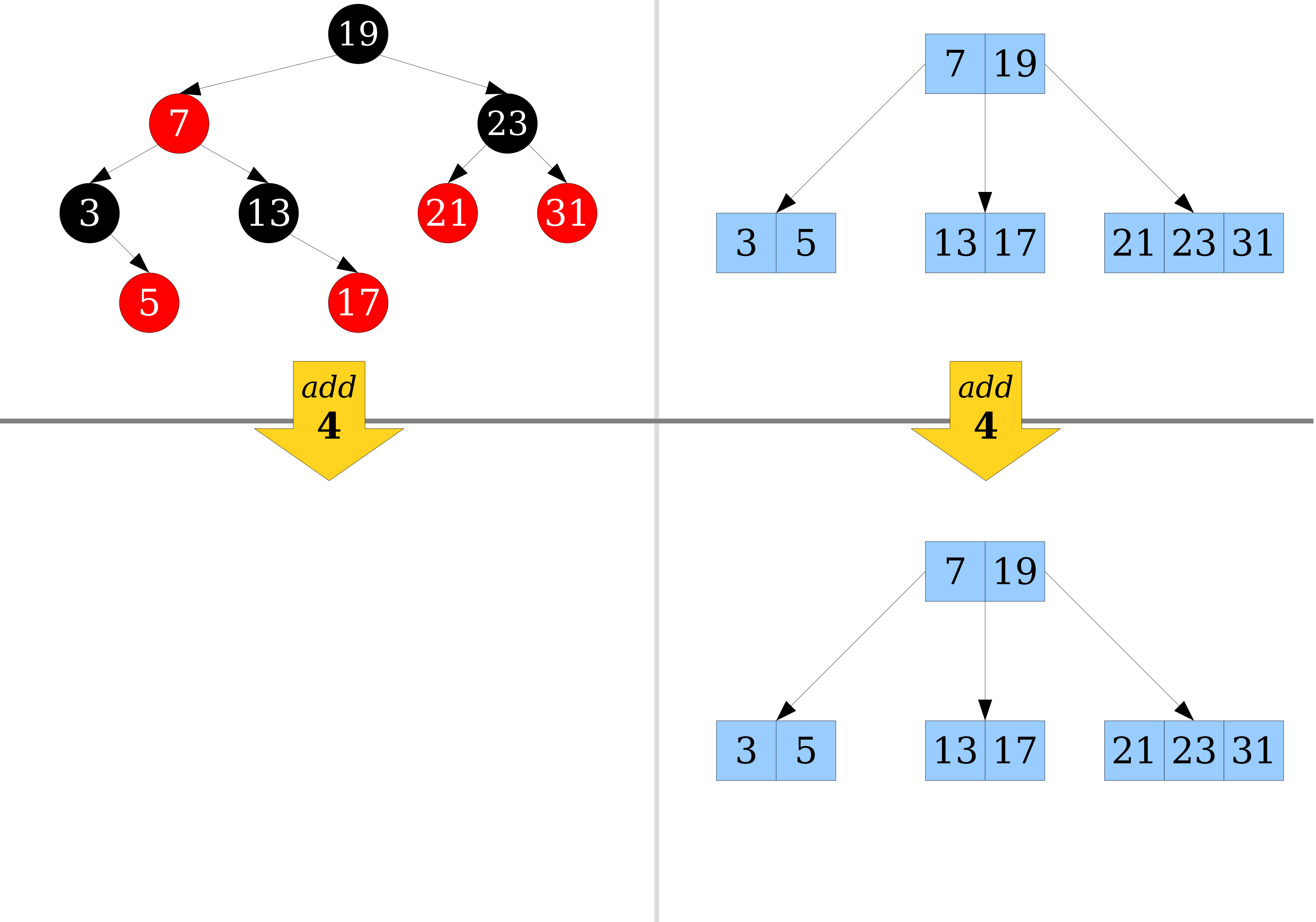


Goal

Red/Black Tree Insertion

- ***Rule #1:*** When inserting a node, if its parent is black, make the node red and stop.
- ***Justification:*** This simulates inserting a key into an existing 2-node or 3-node.

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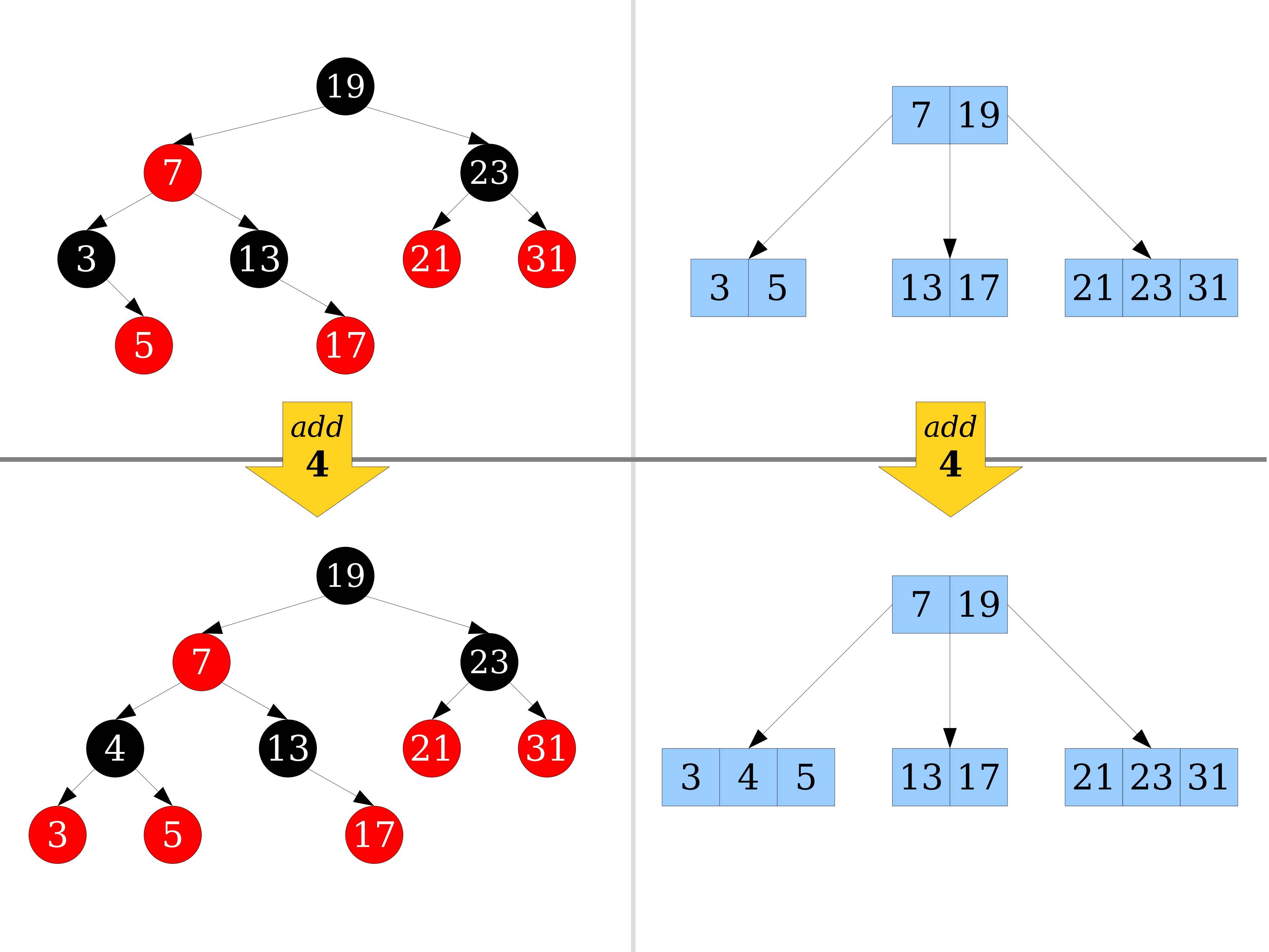
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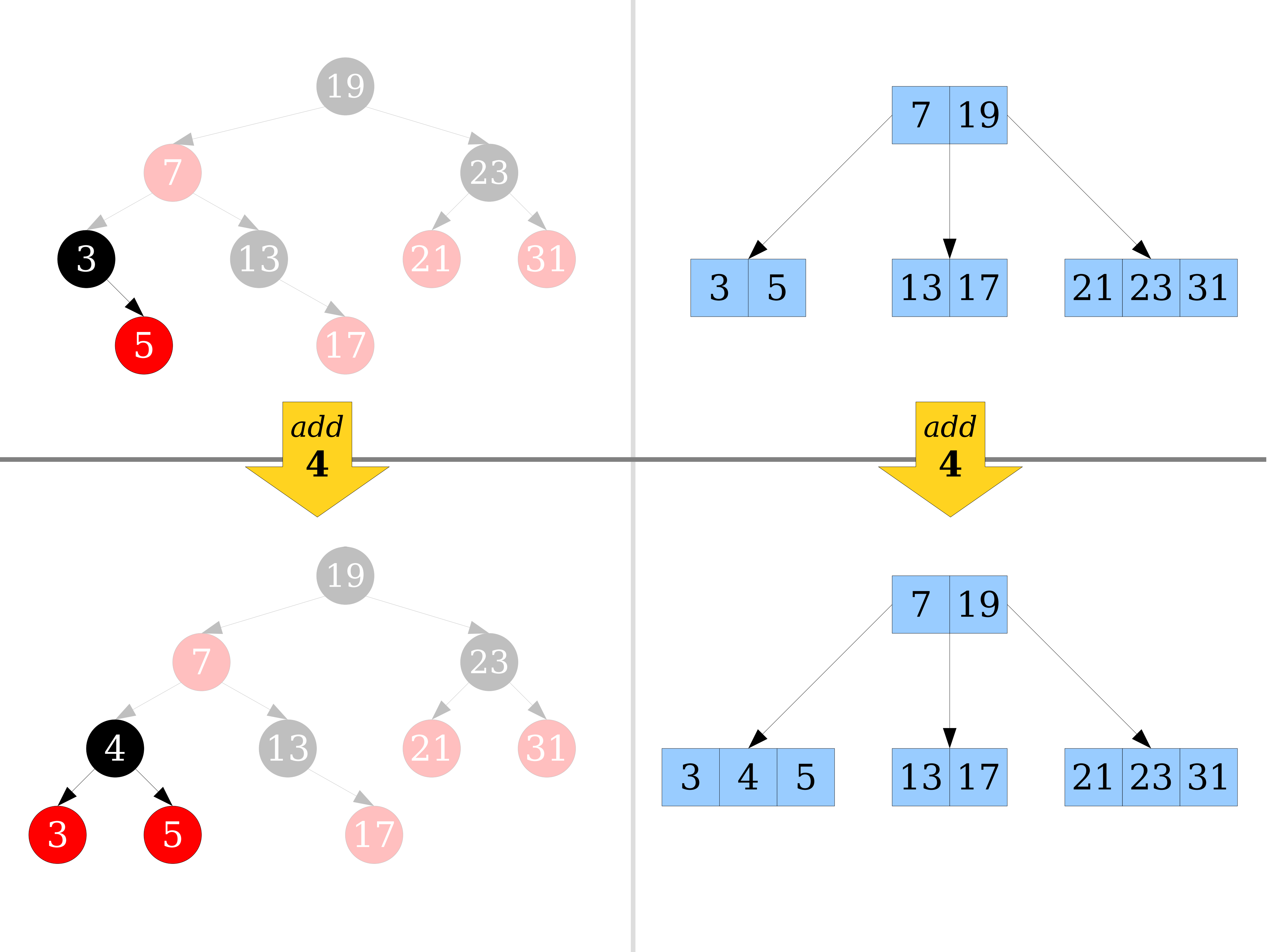
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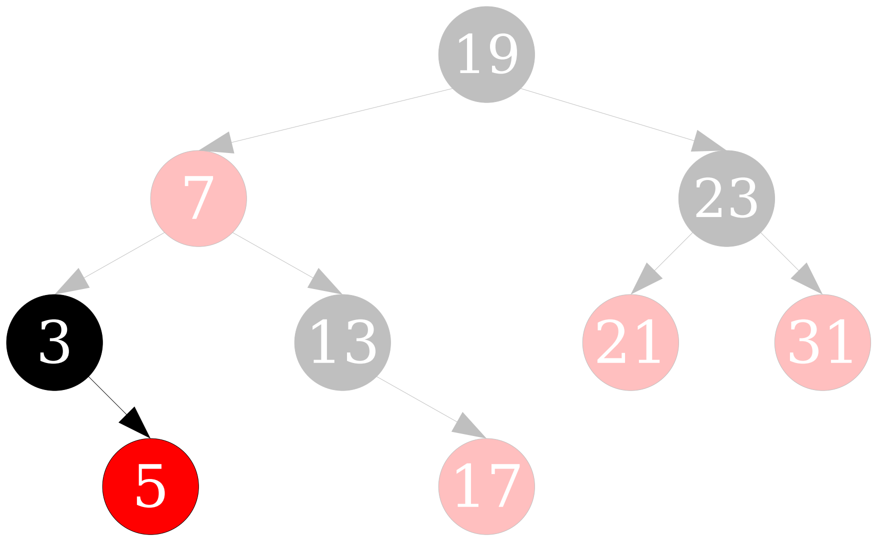

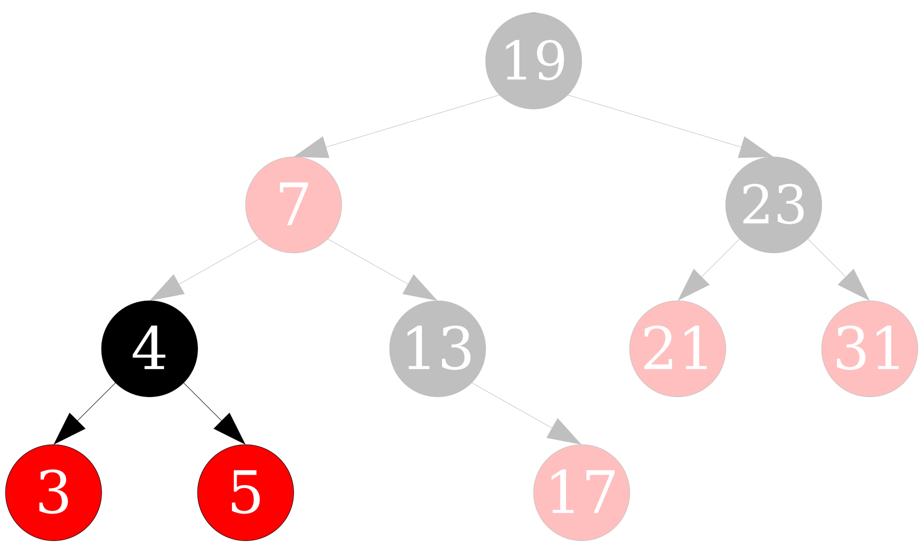

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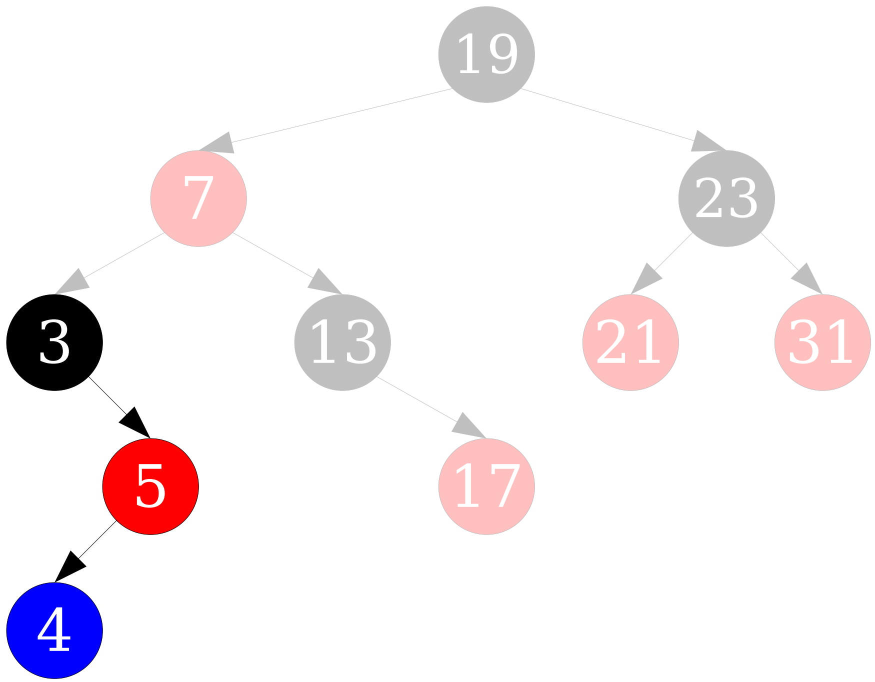
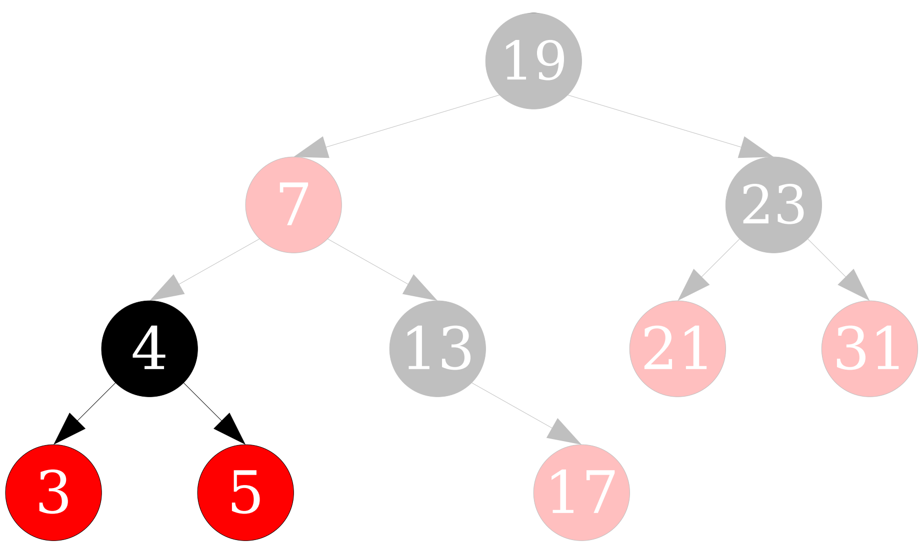


We need to move nodes around in a binary search tree. How do we do this?



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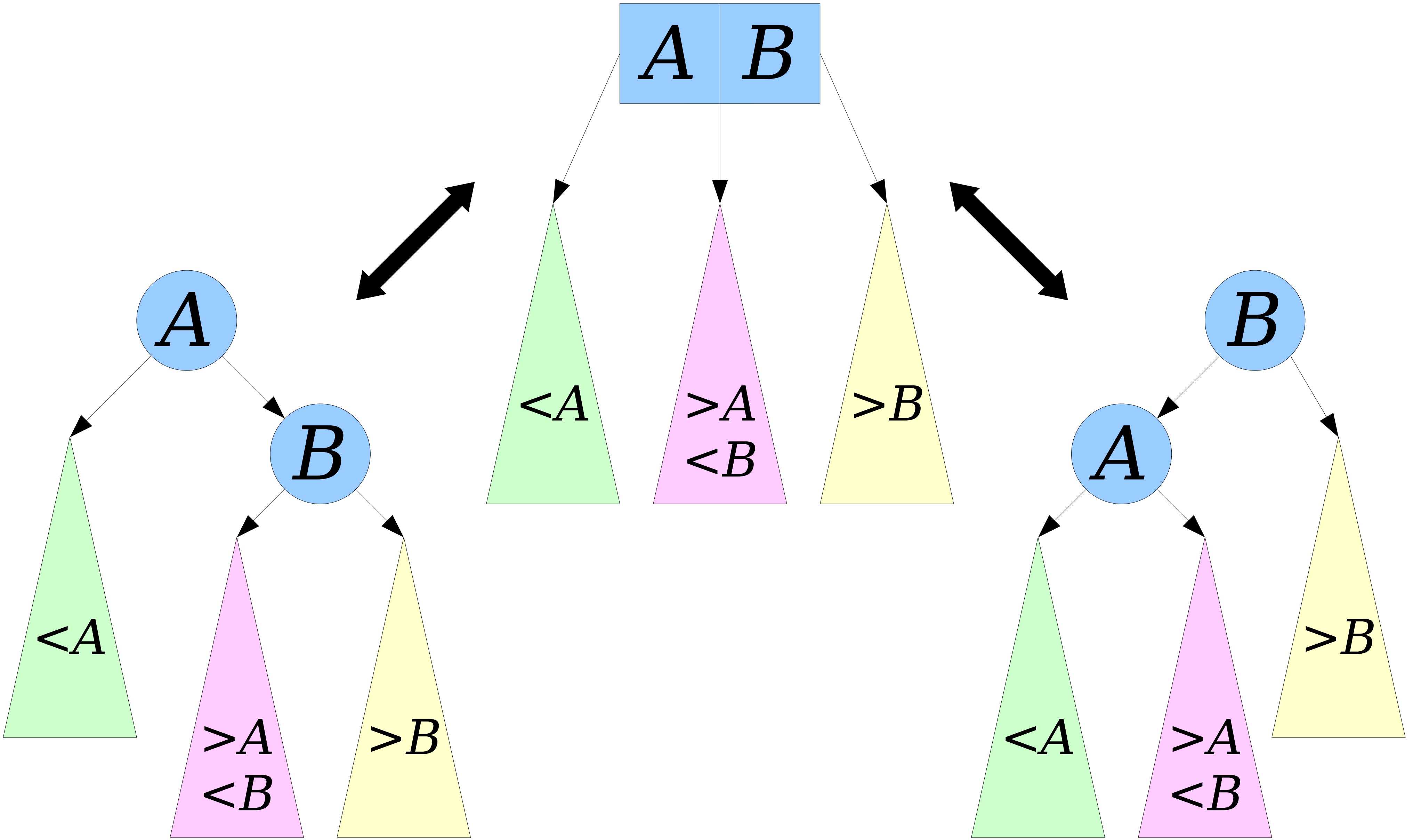
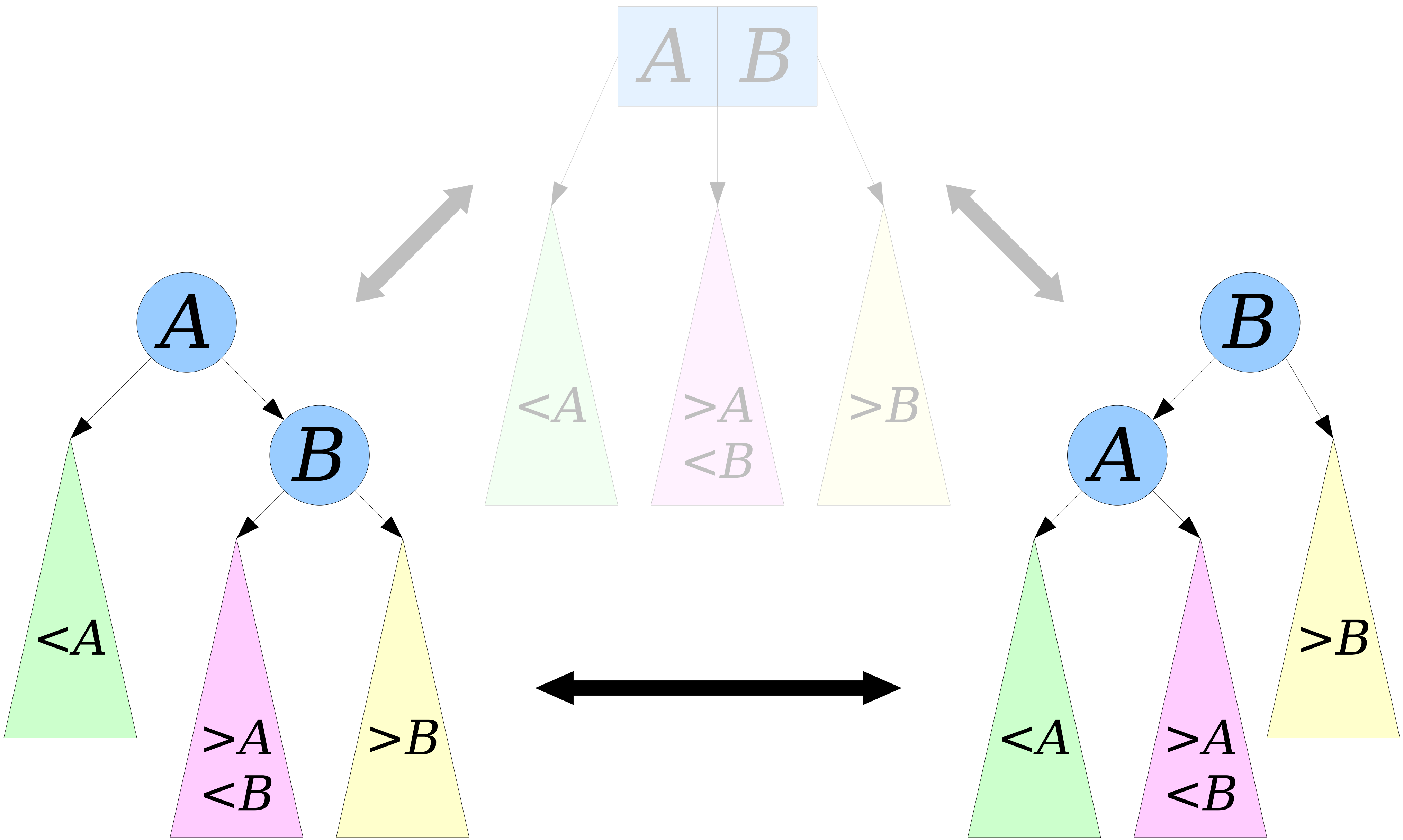
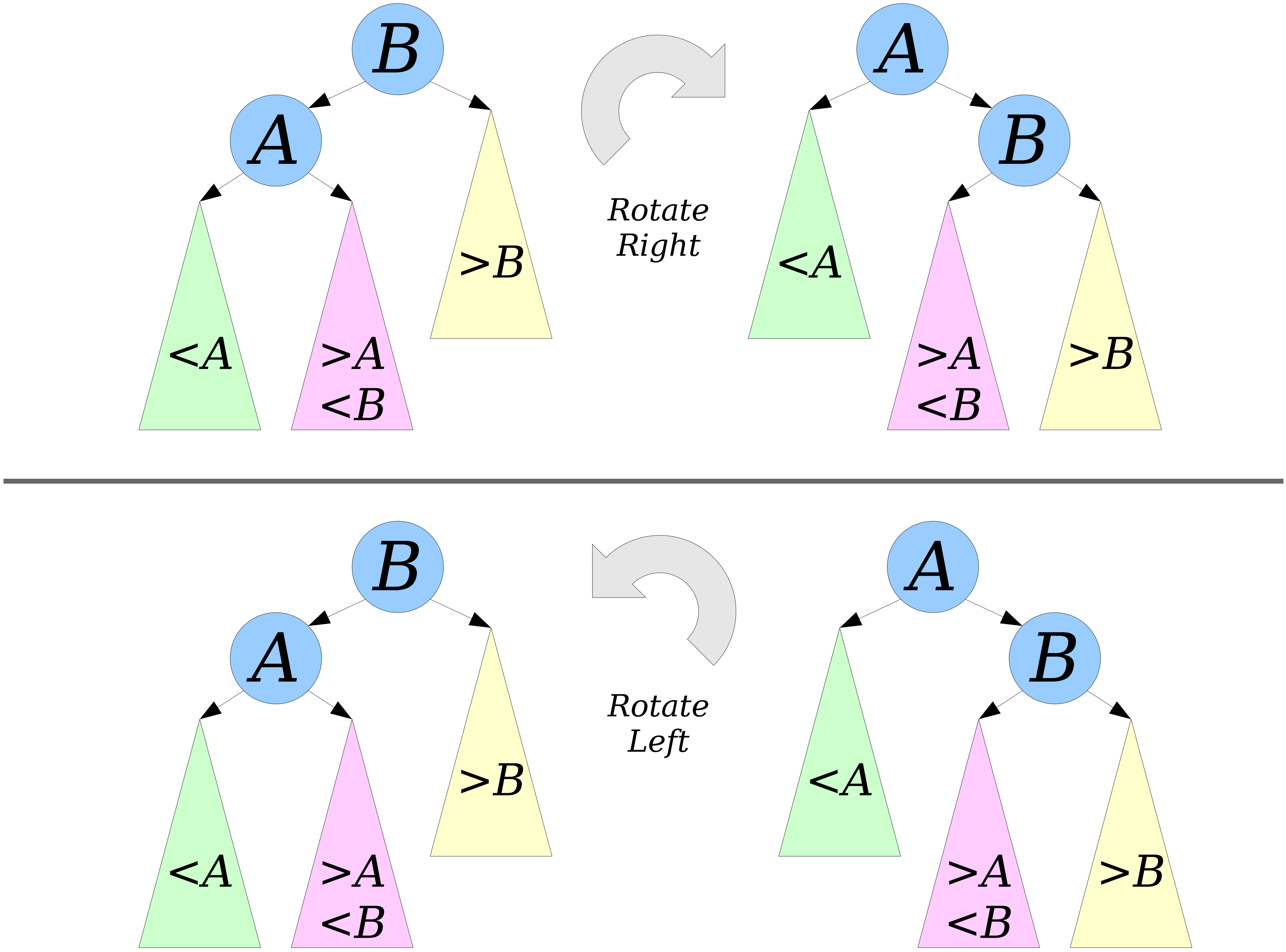
Tree Rotations



Tree Rotations

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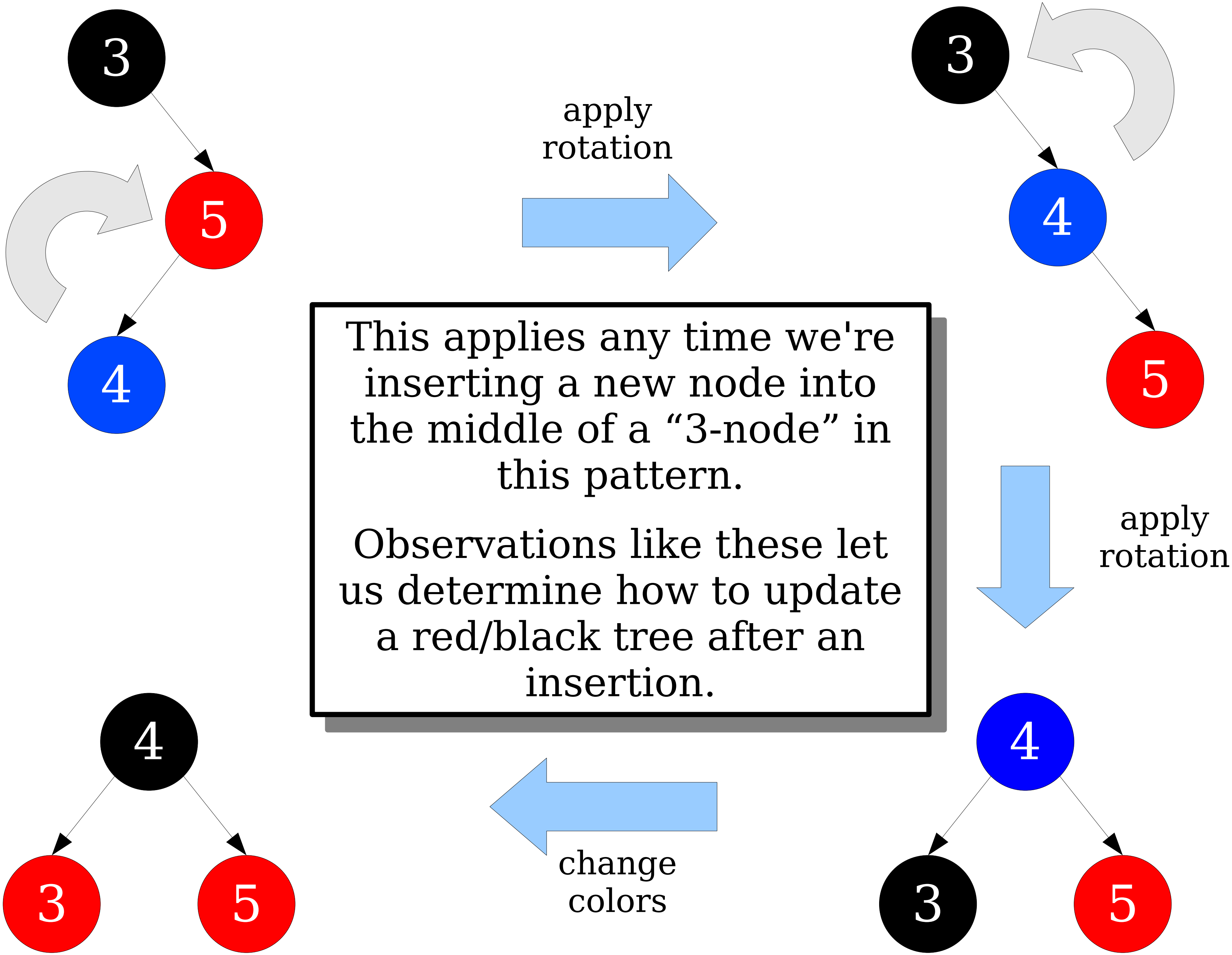
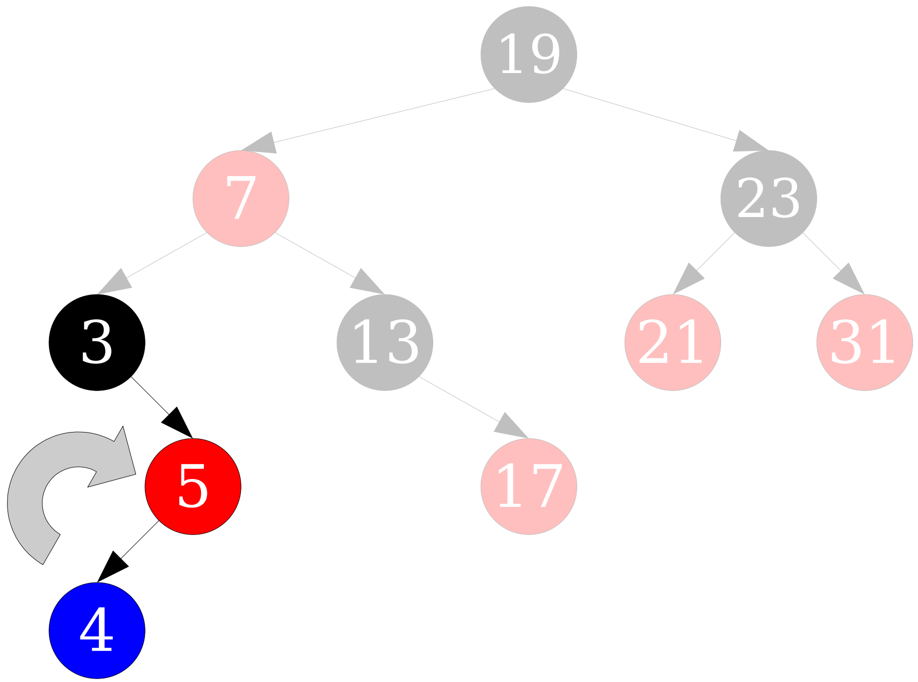
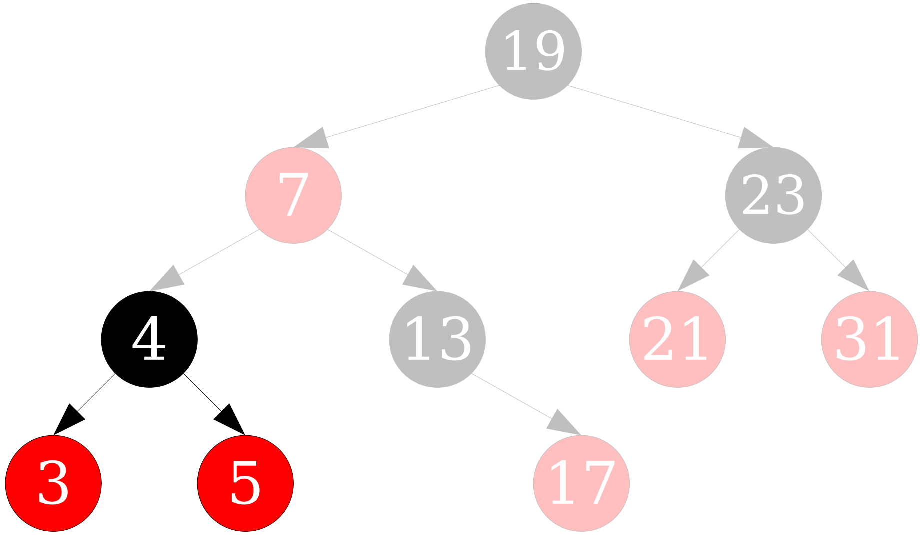


This applies any time we're inserting a new node into the middle of a “3-node” in this pattern.

Observations like these let us determine how to update a red/black tree after an insertion.





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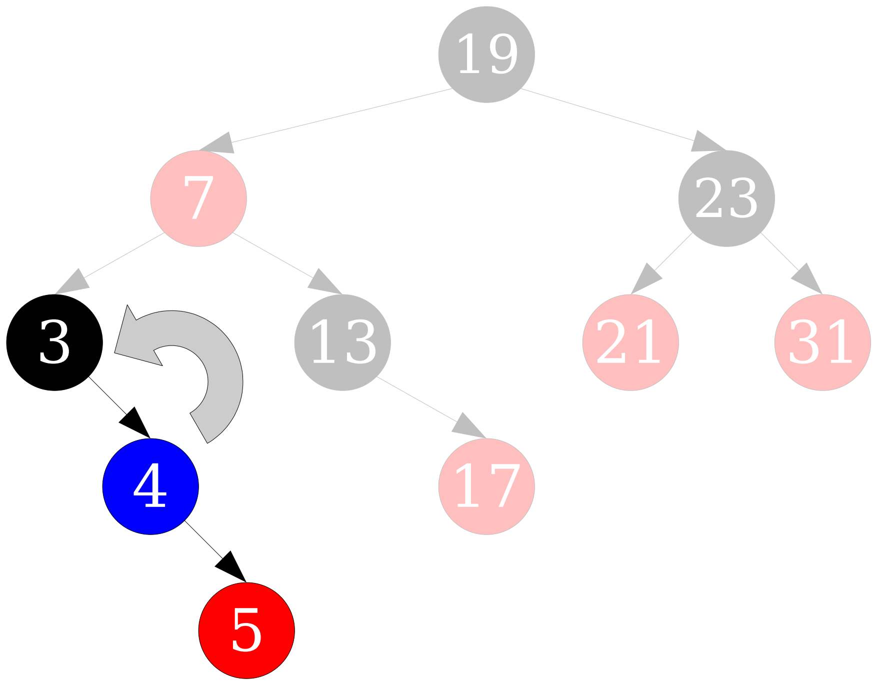

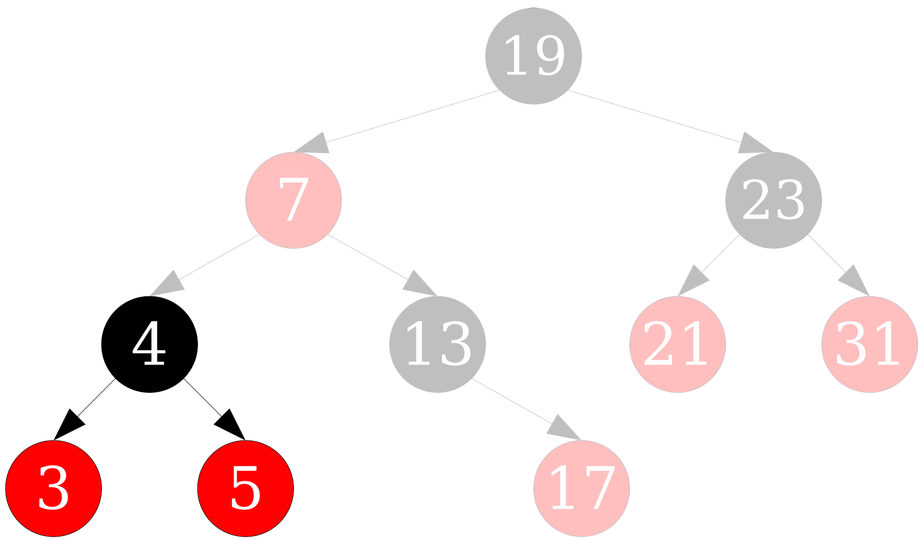

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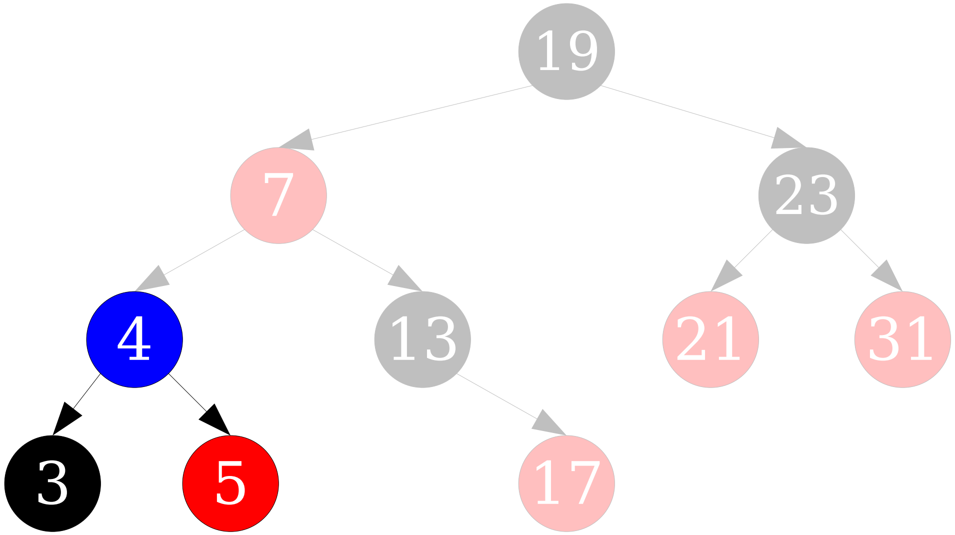

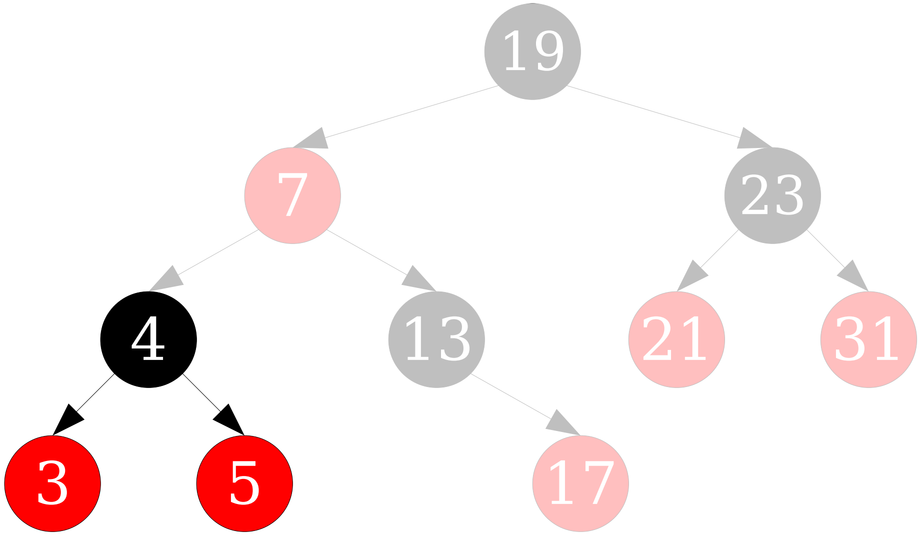

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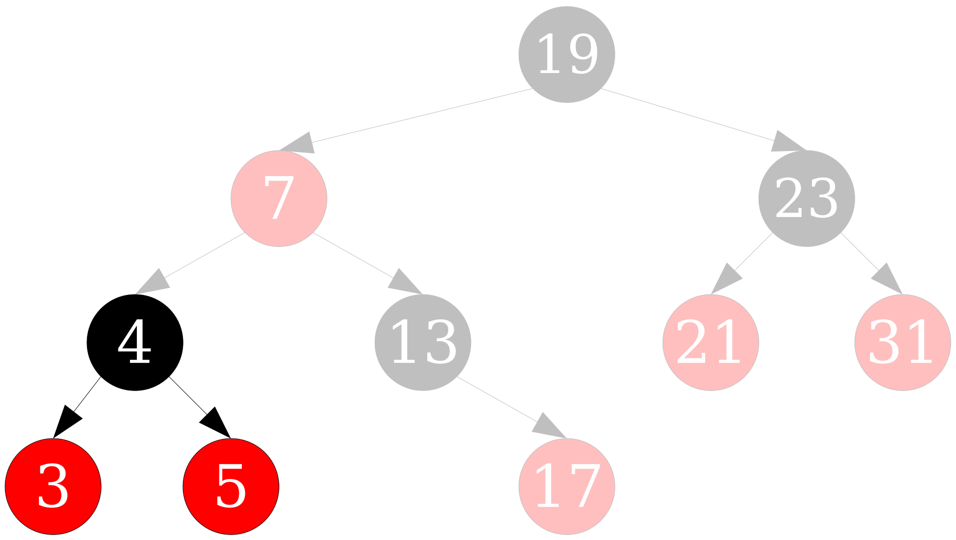

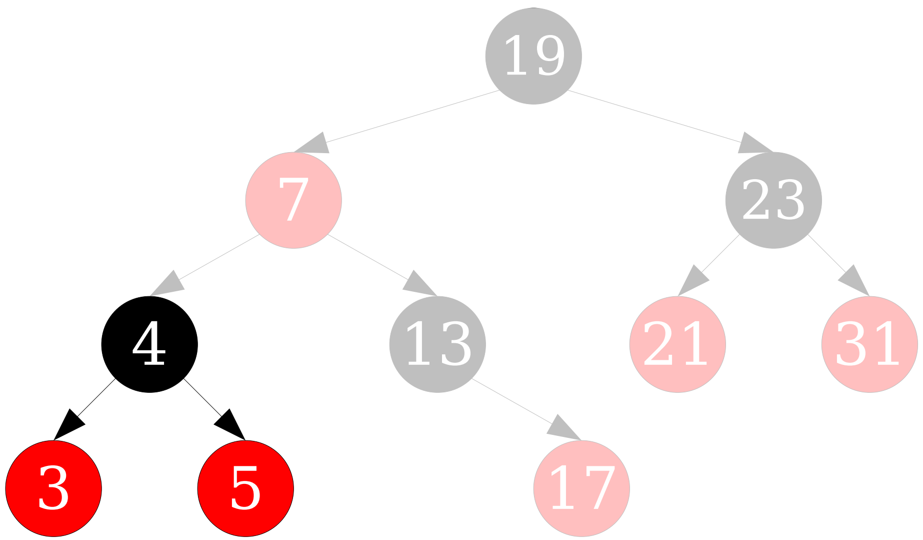

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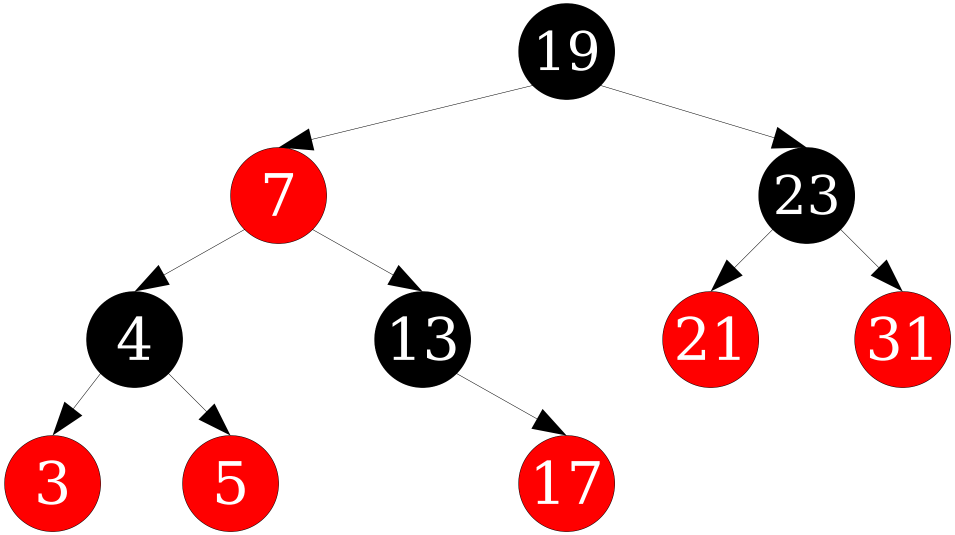

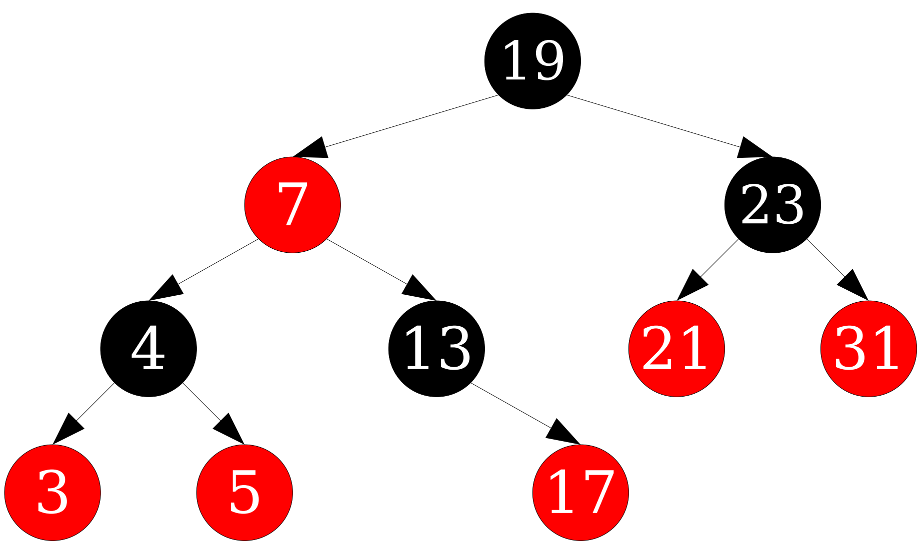

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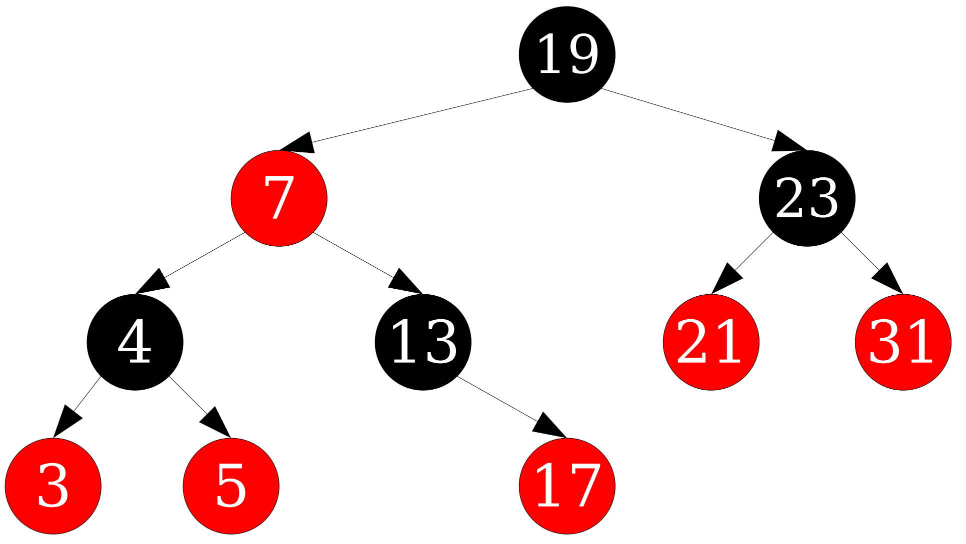
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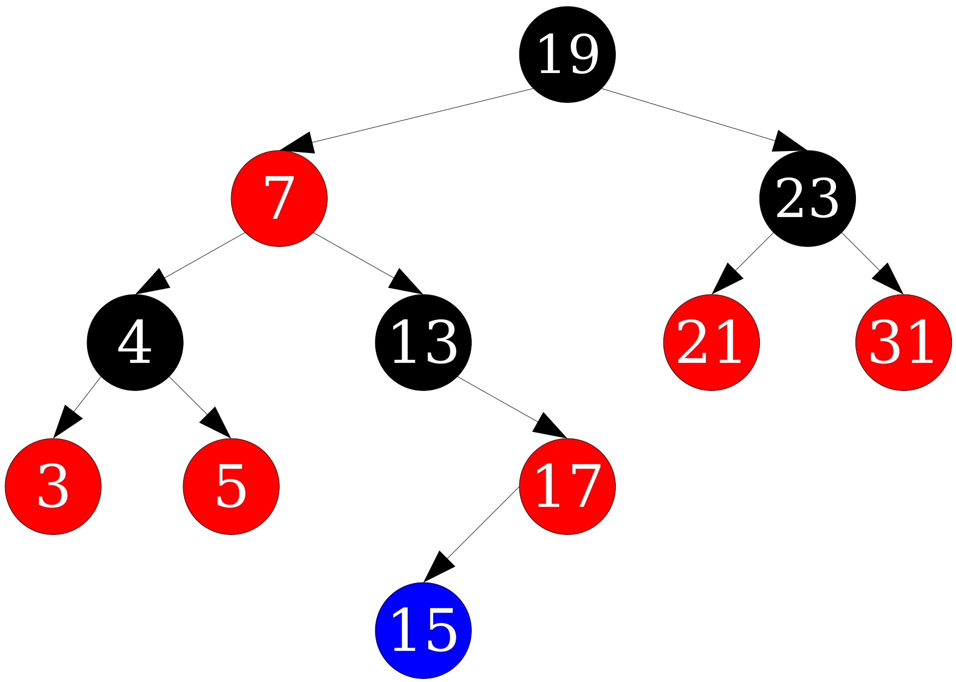
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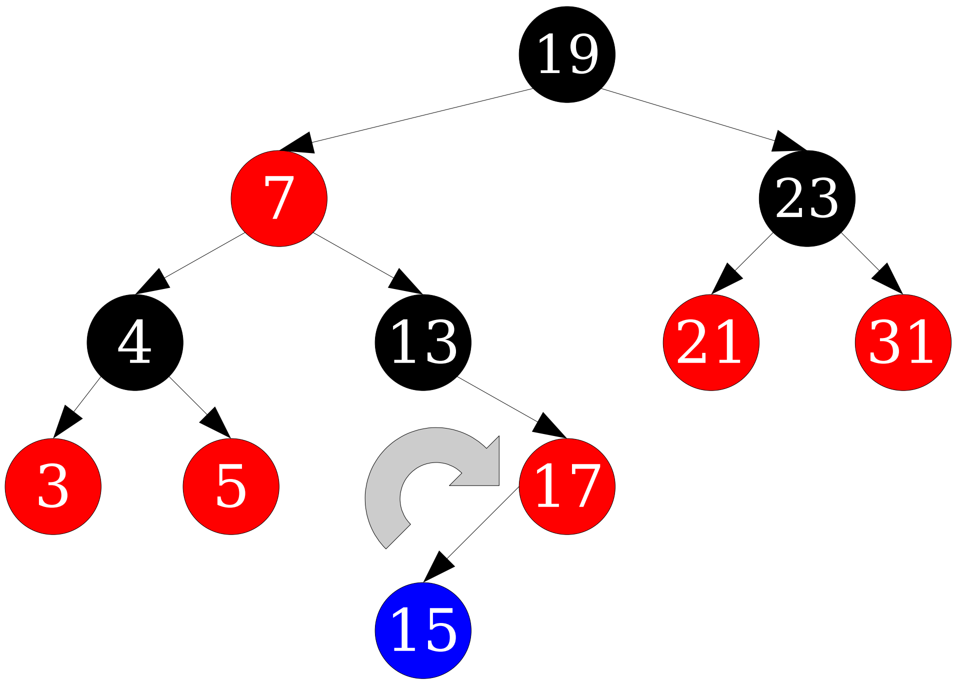
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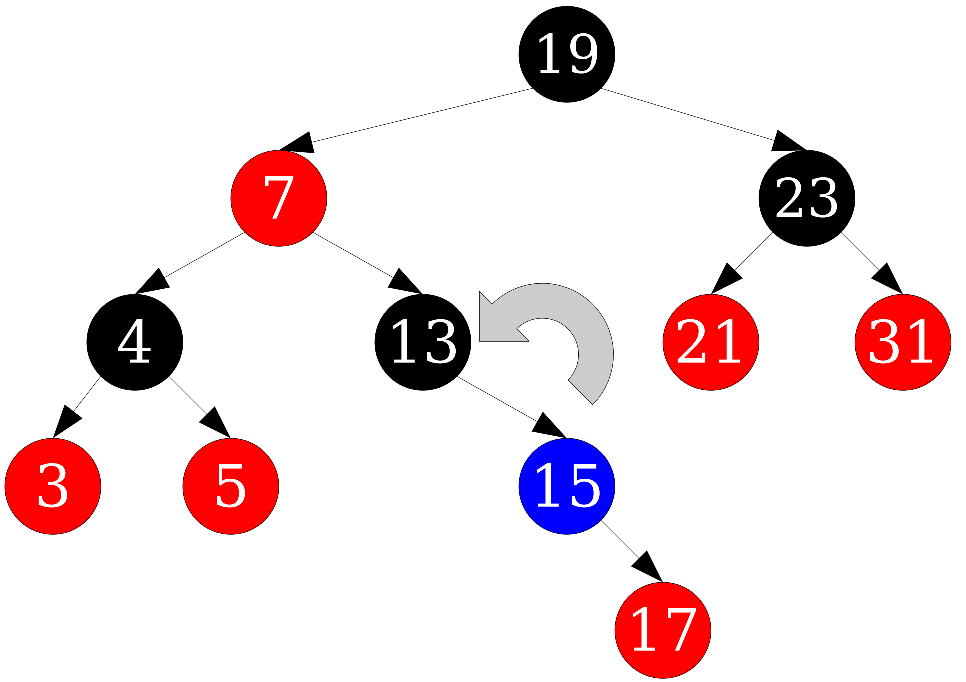
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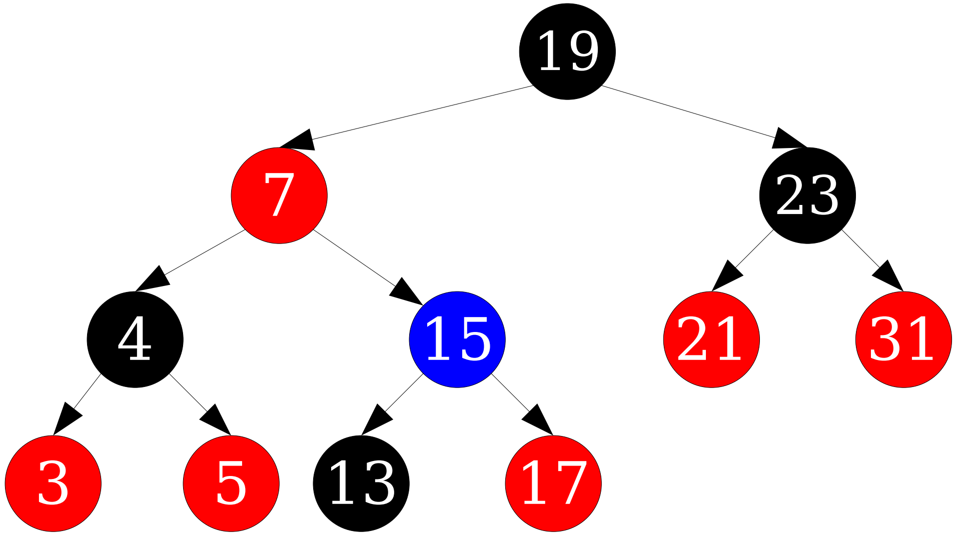
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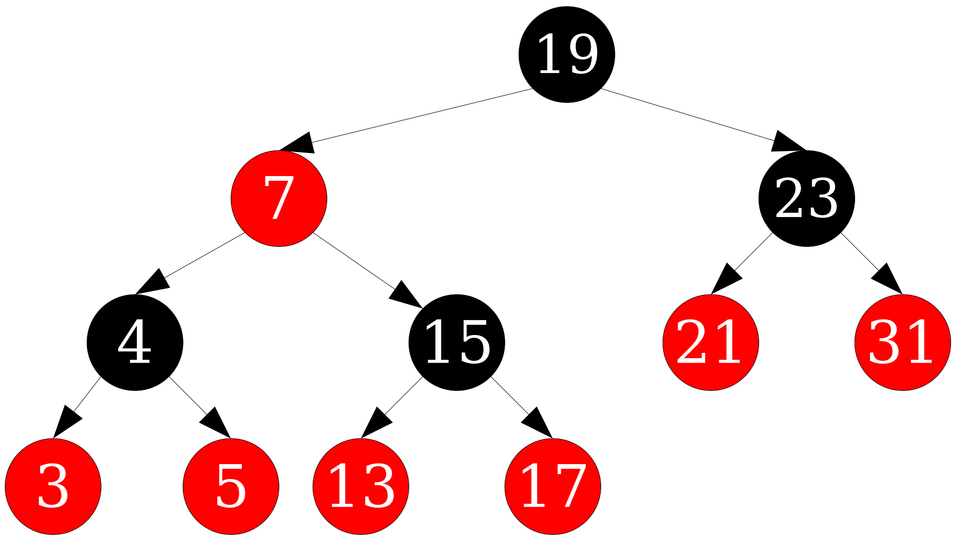
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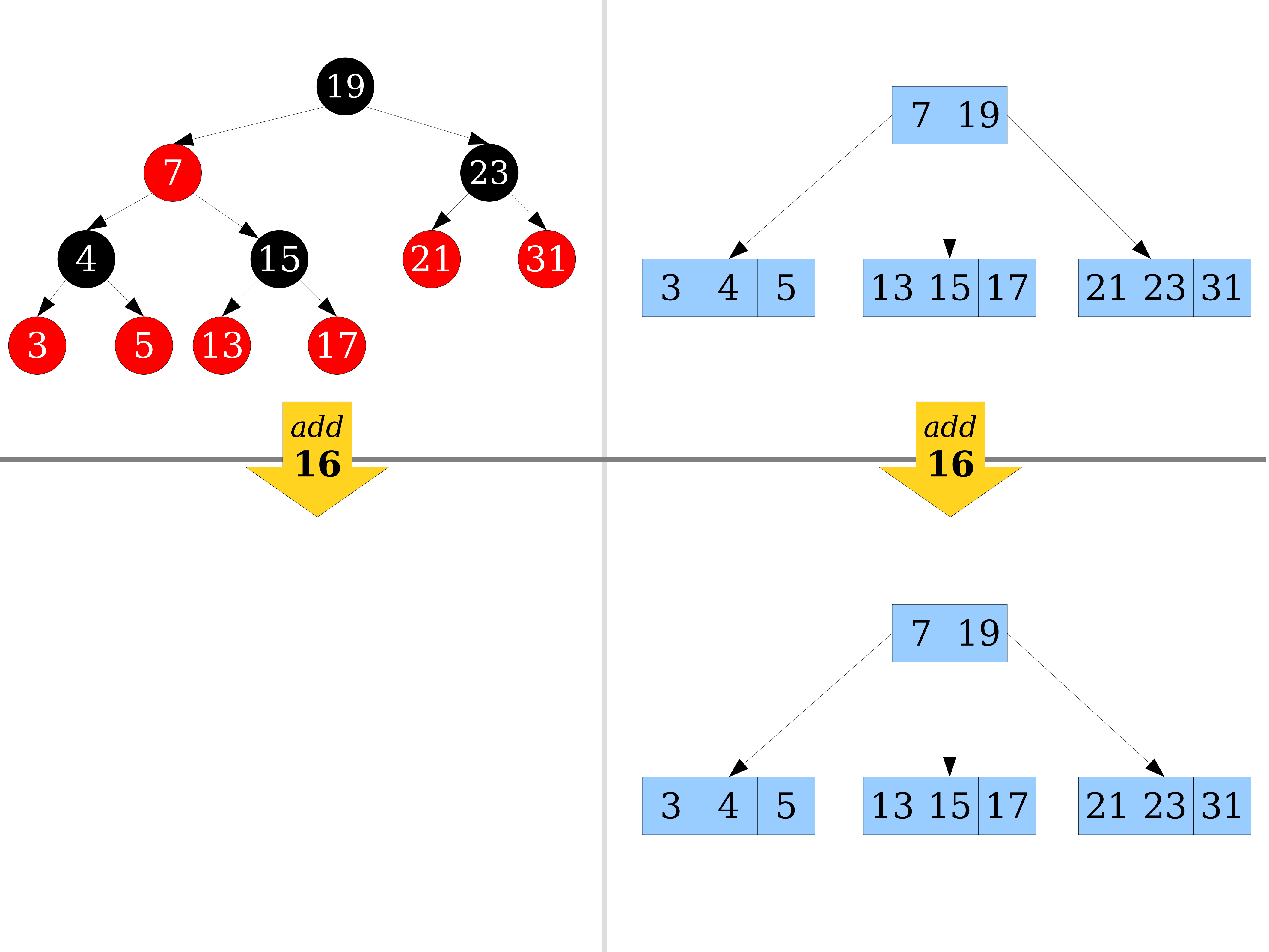
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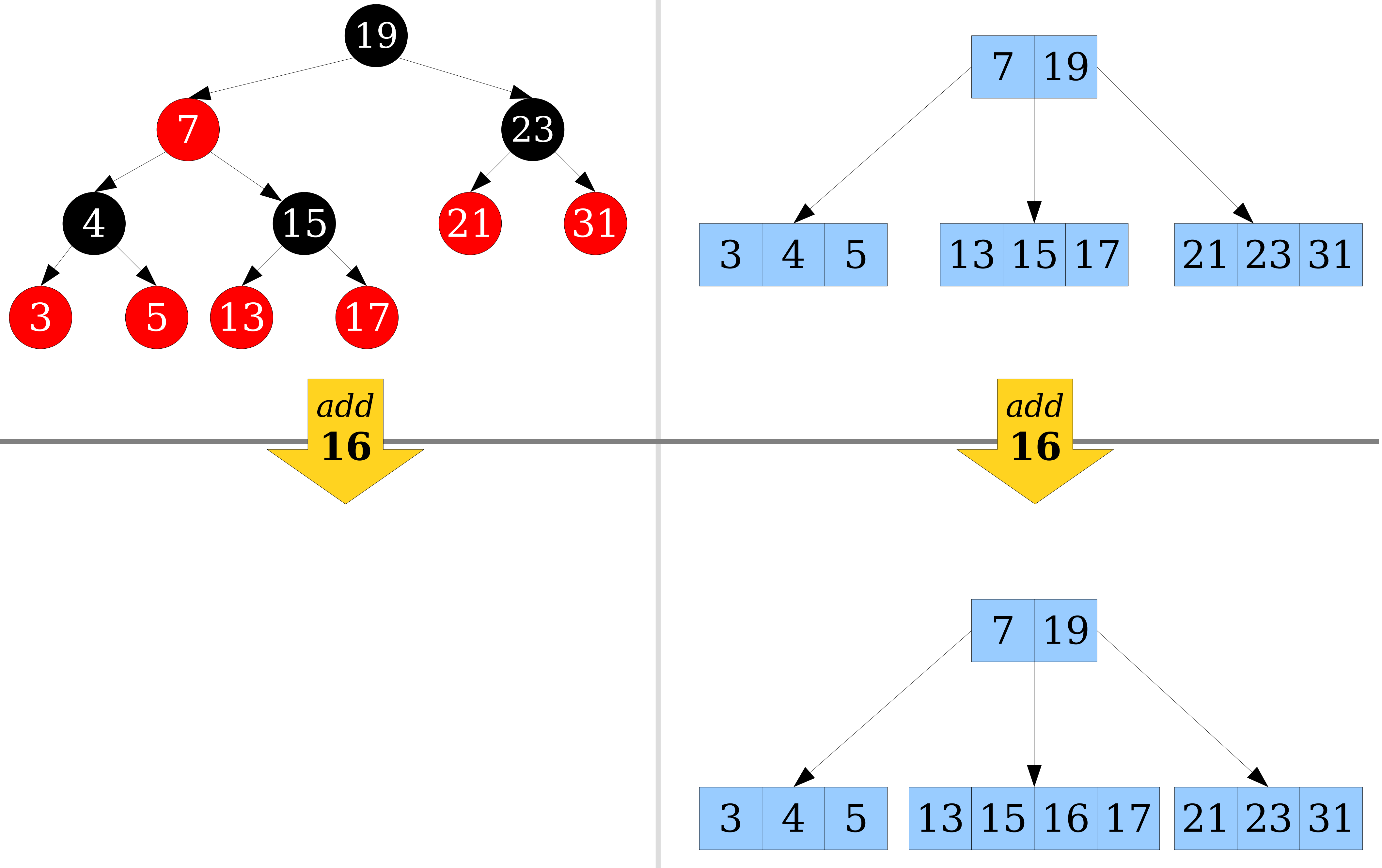
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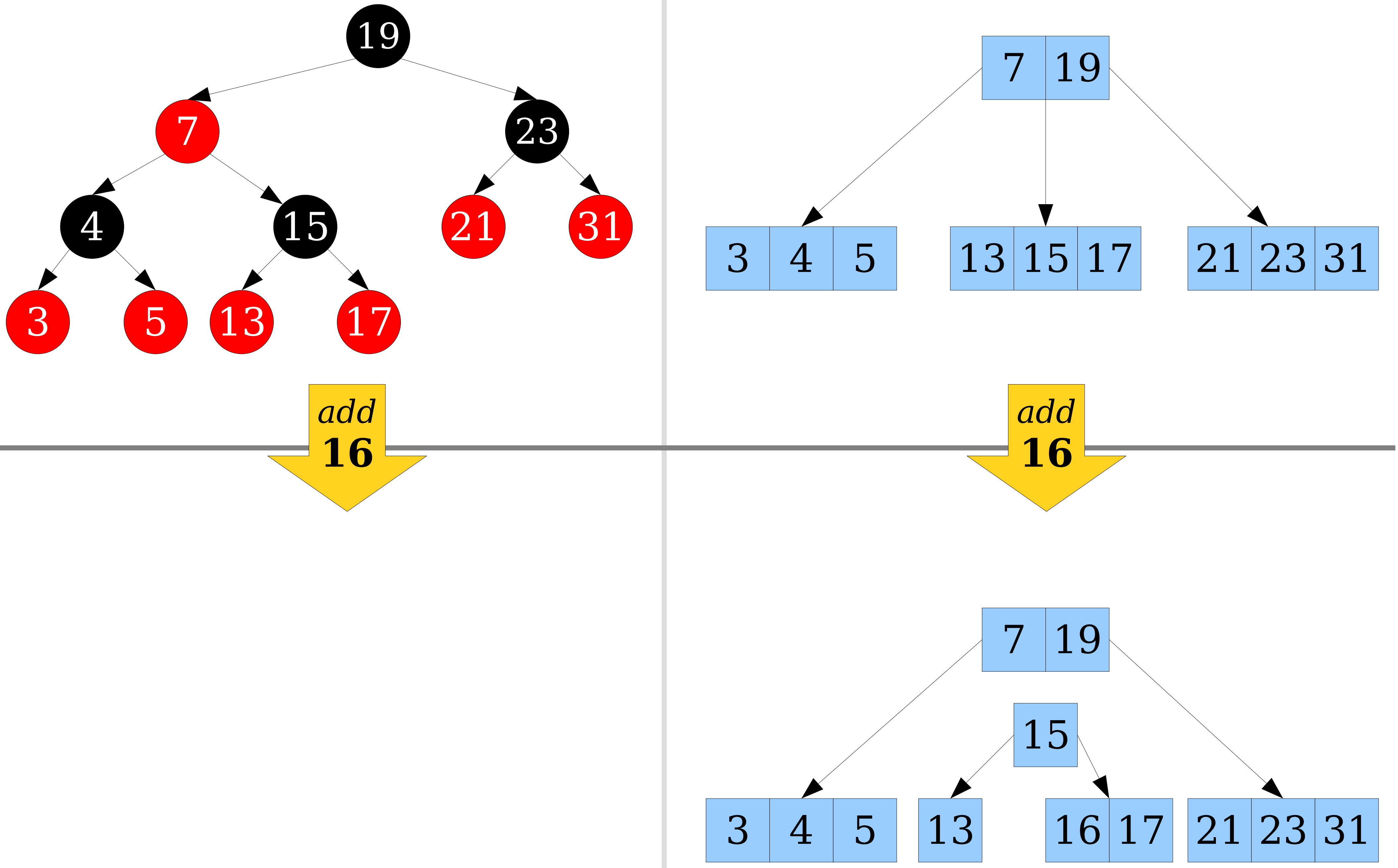

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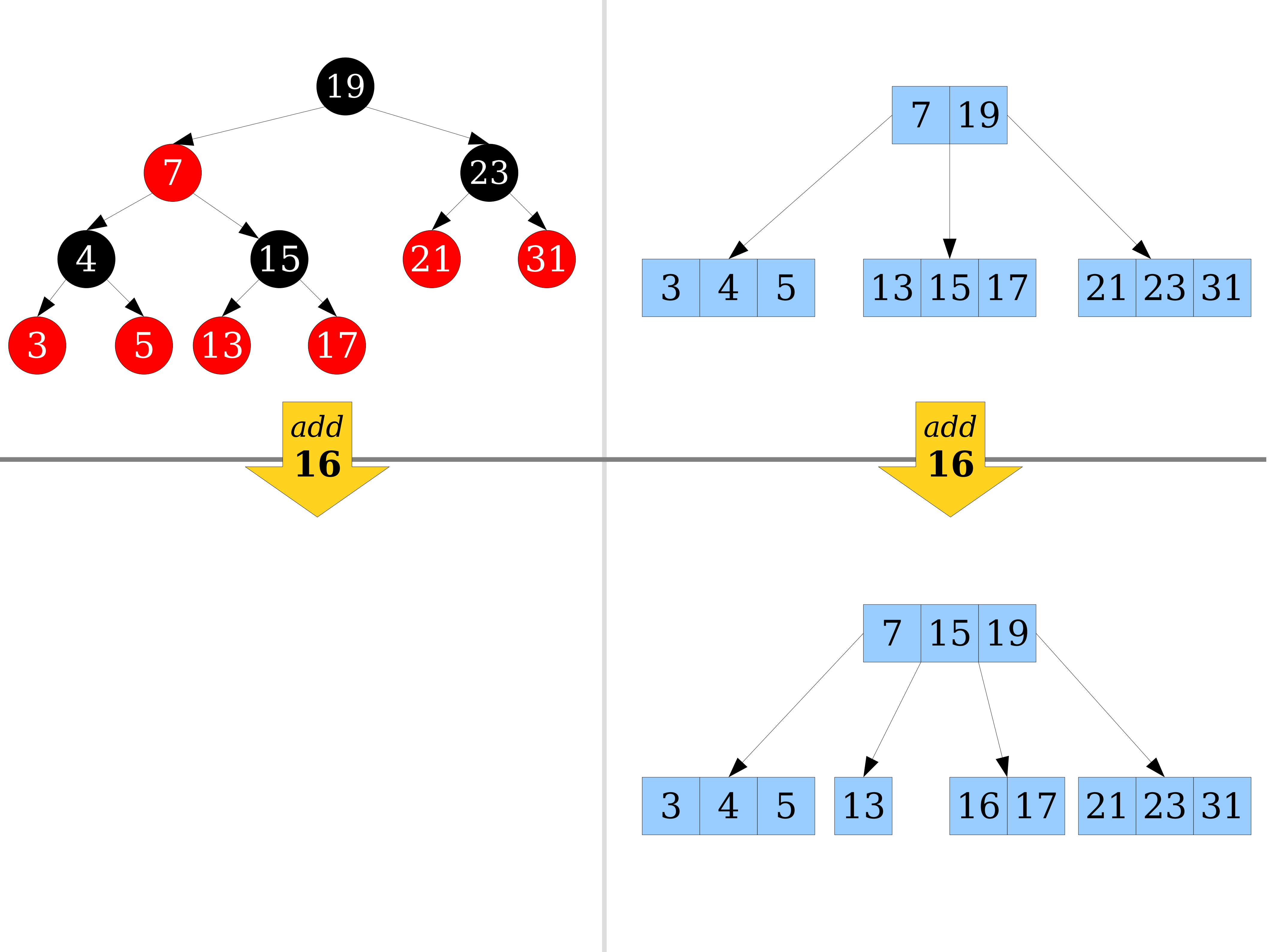

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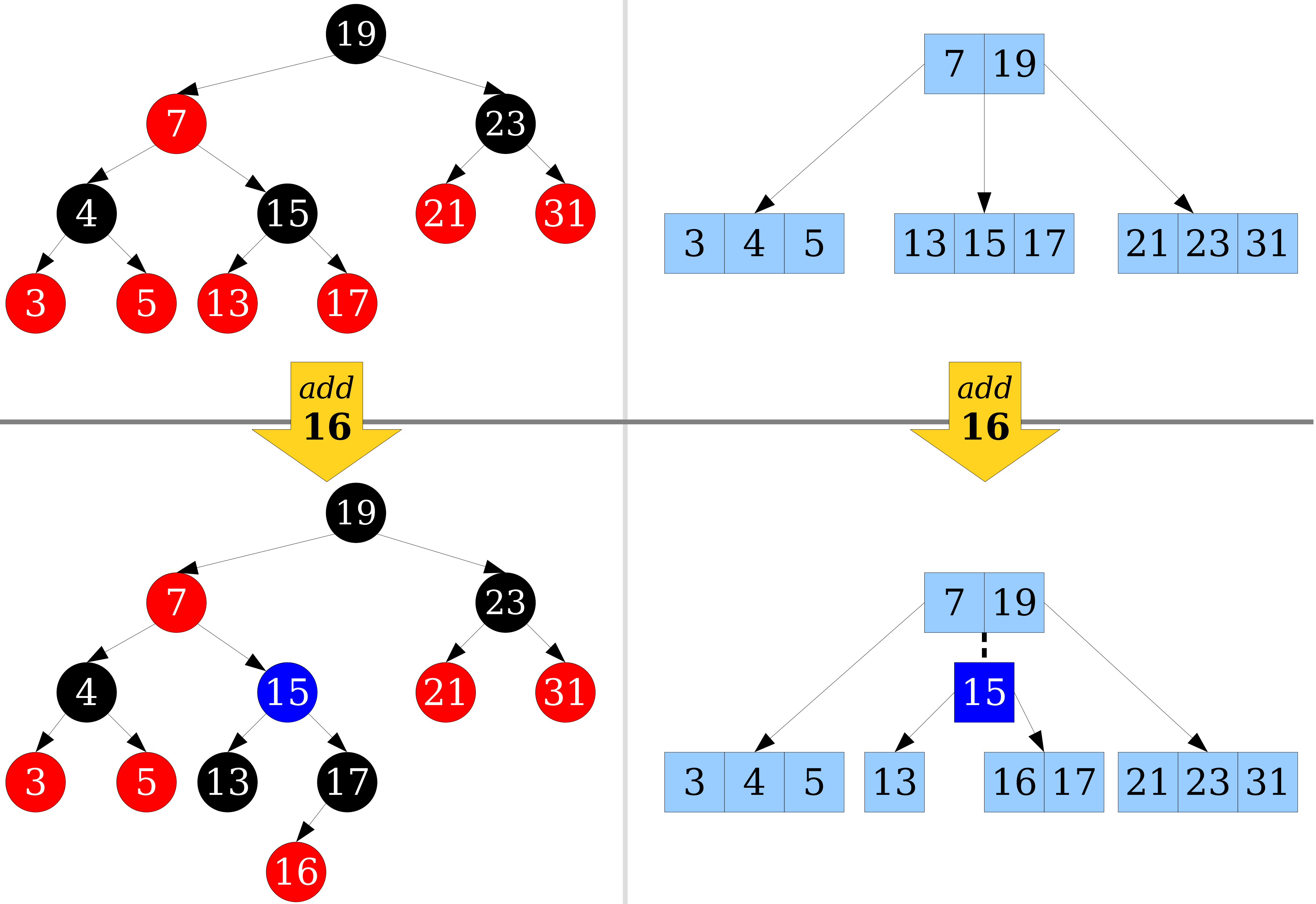

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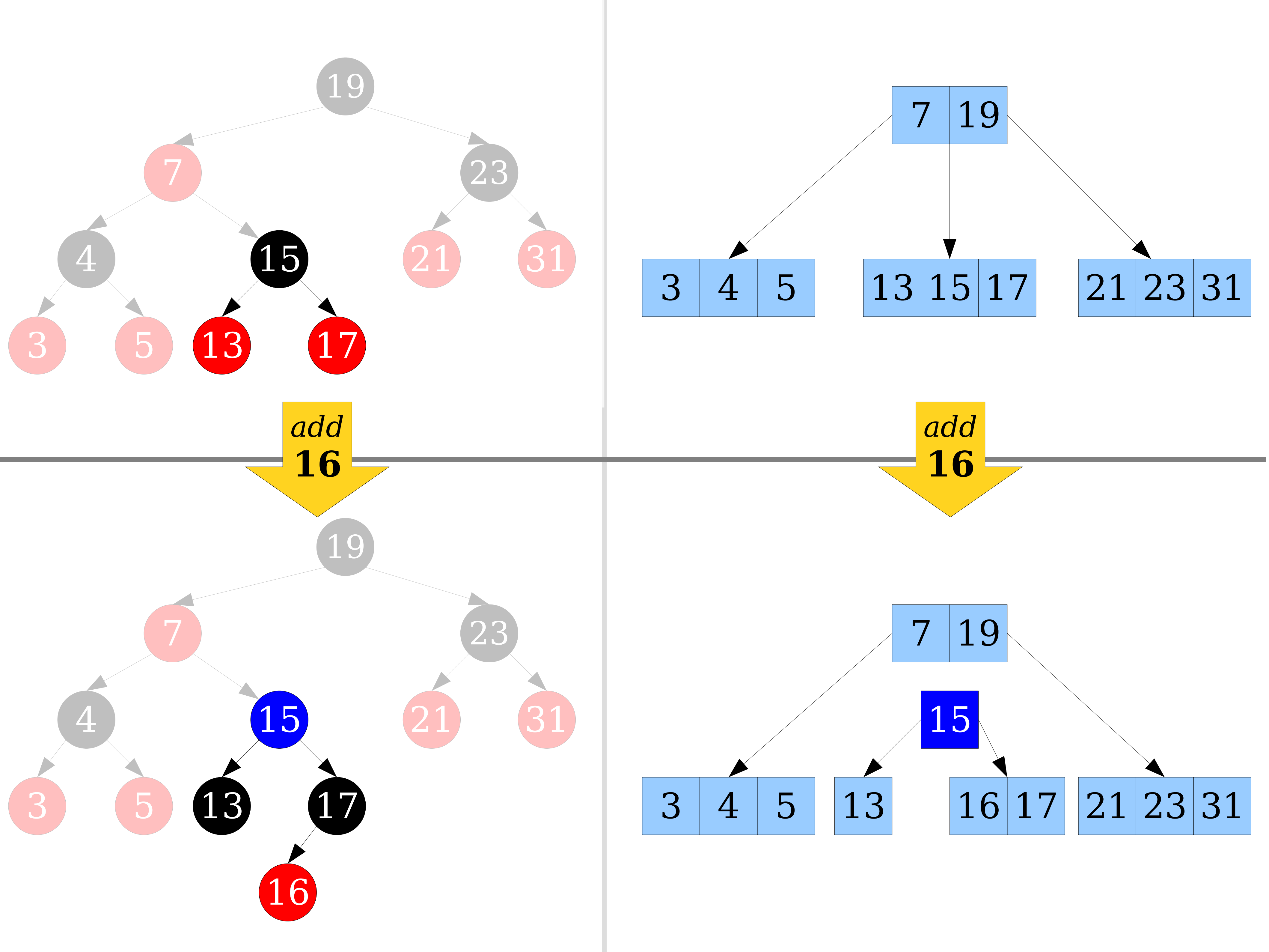
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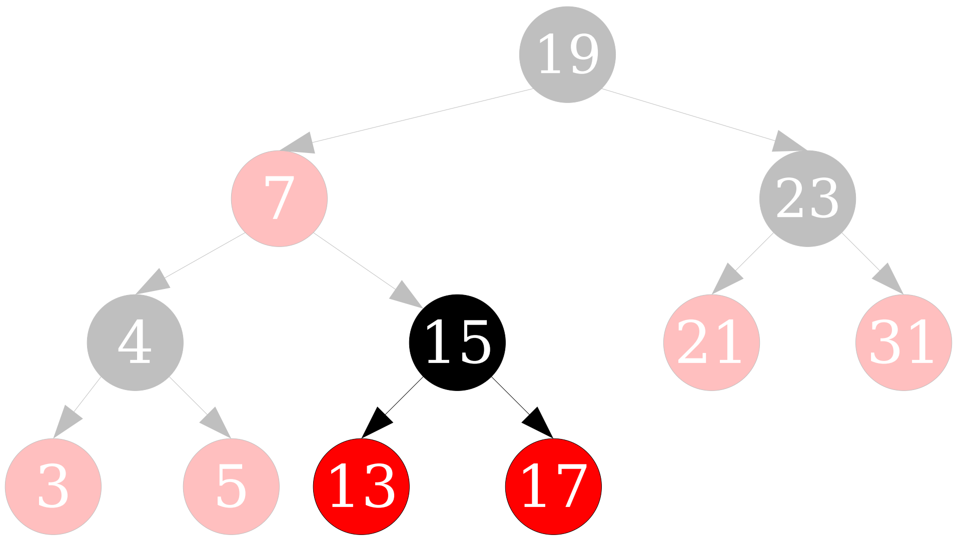
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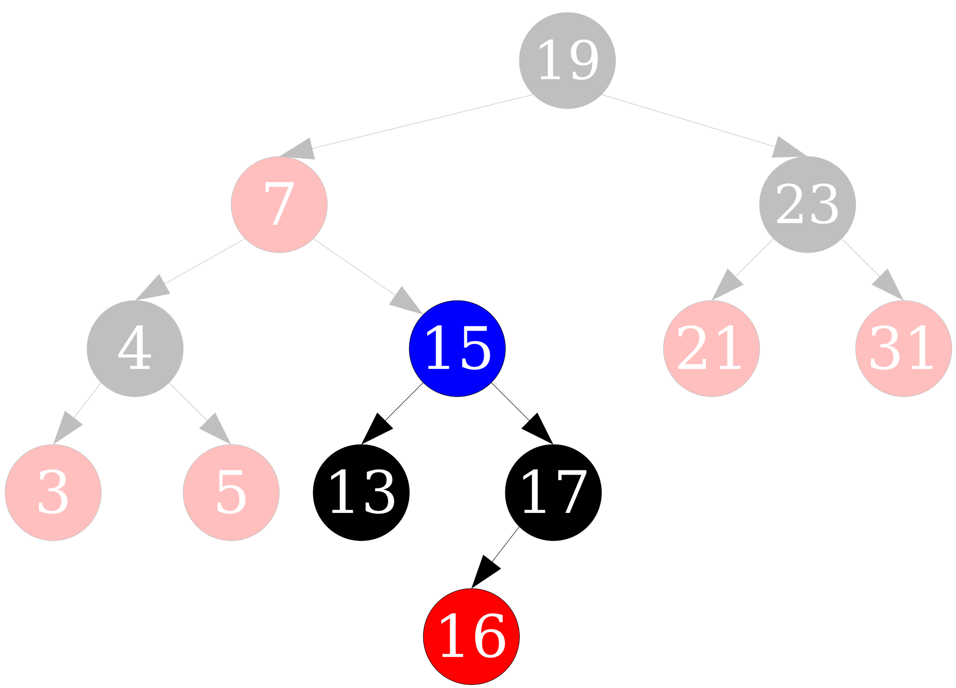
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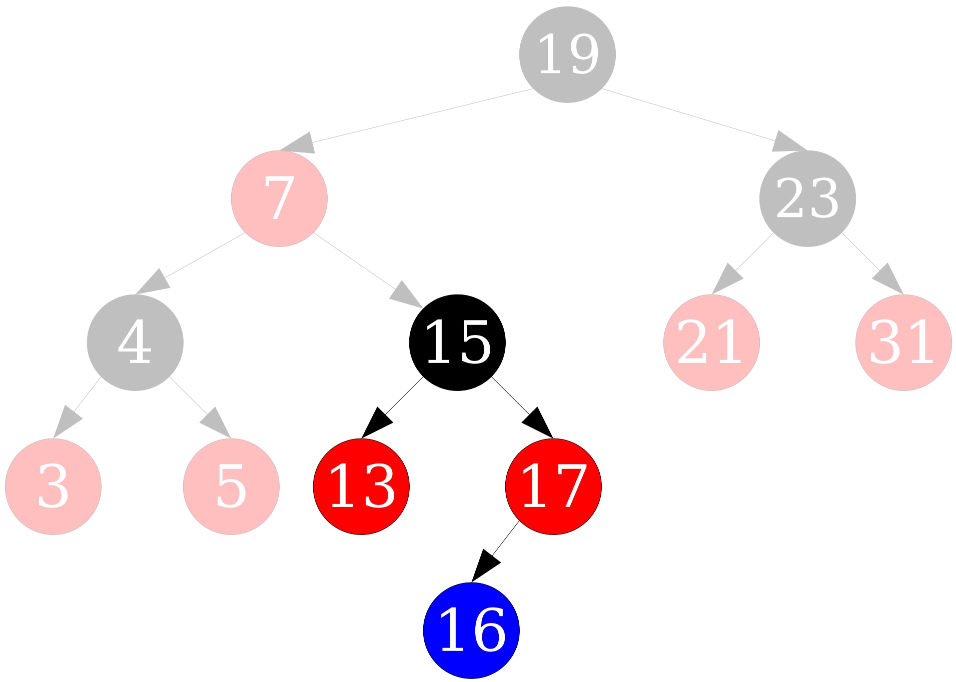
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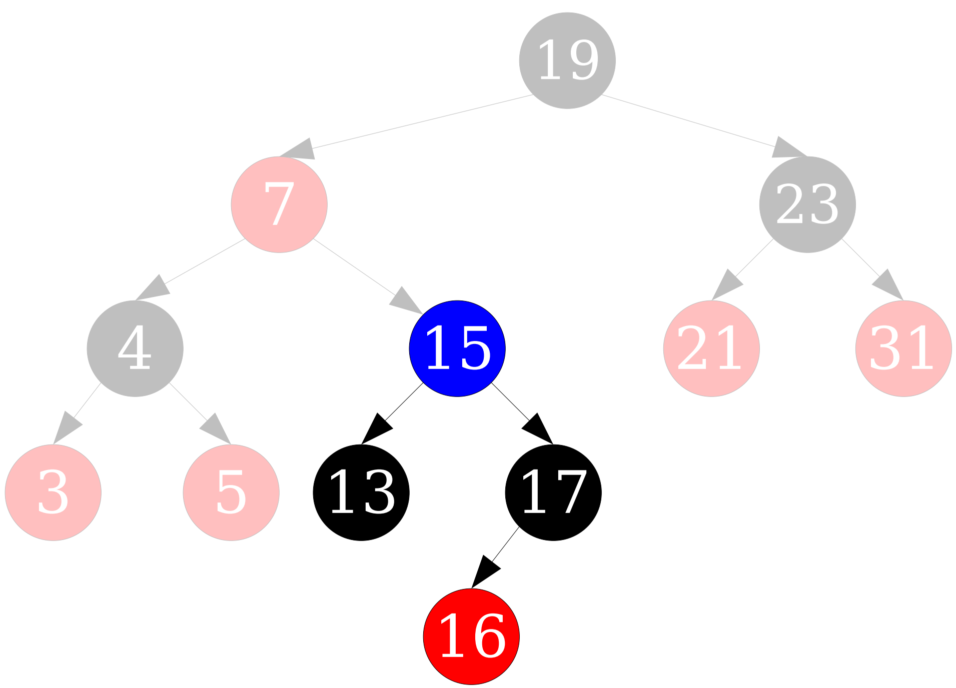
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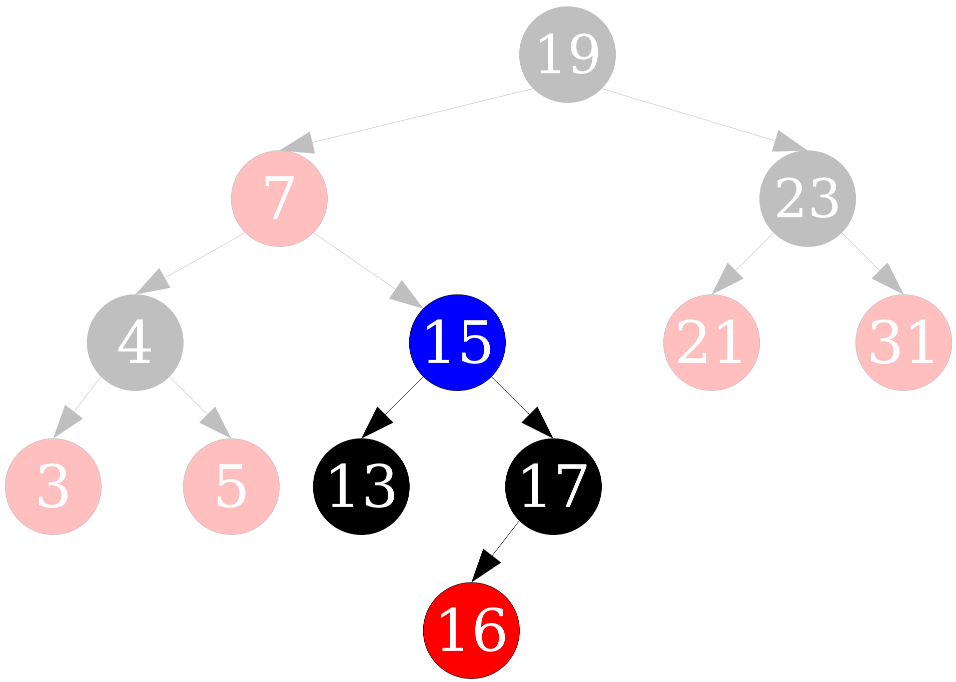
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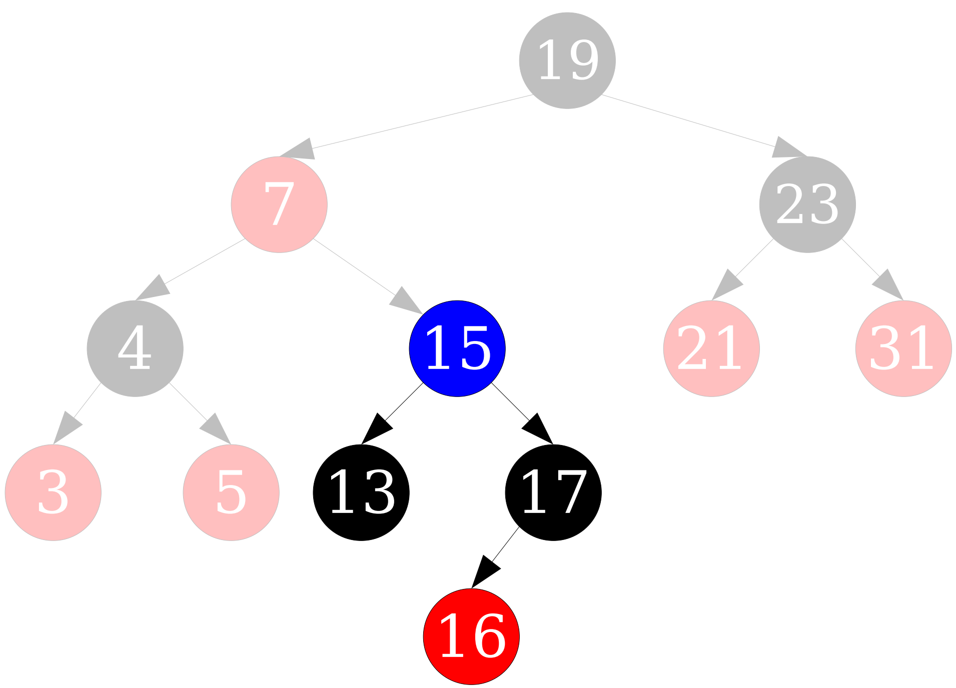
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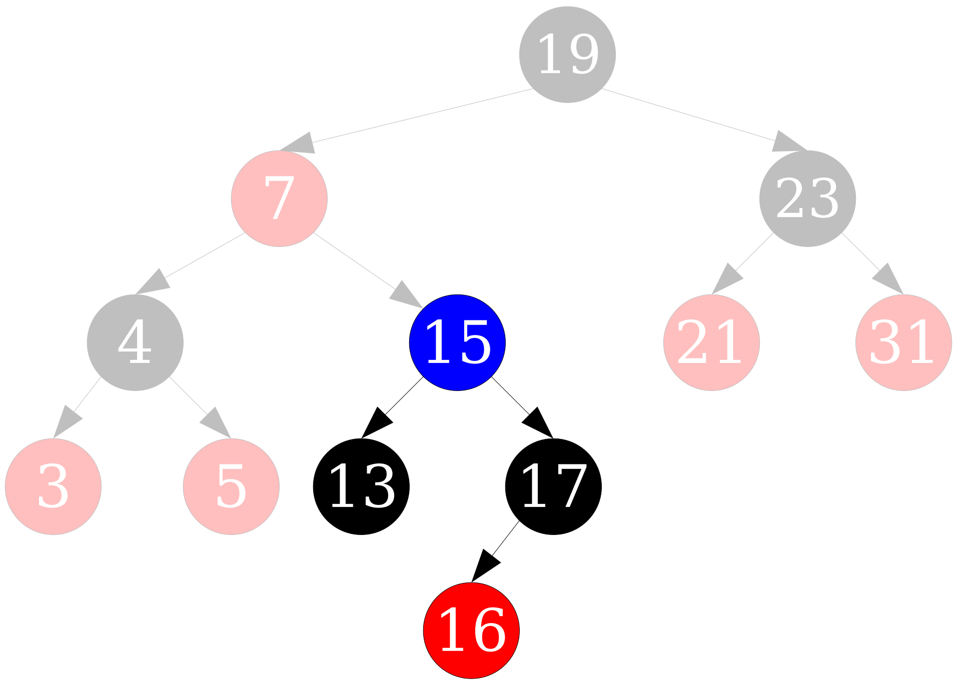
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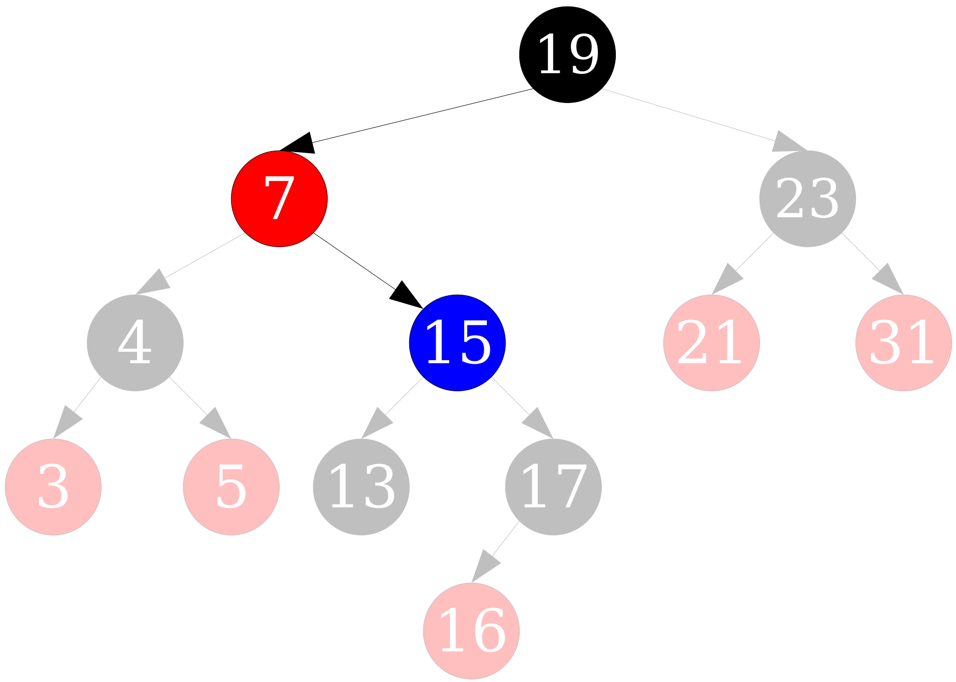
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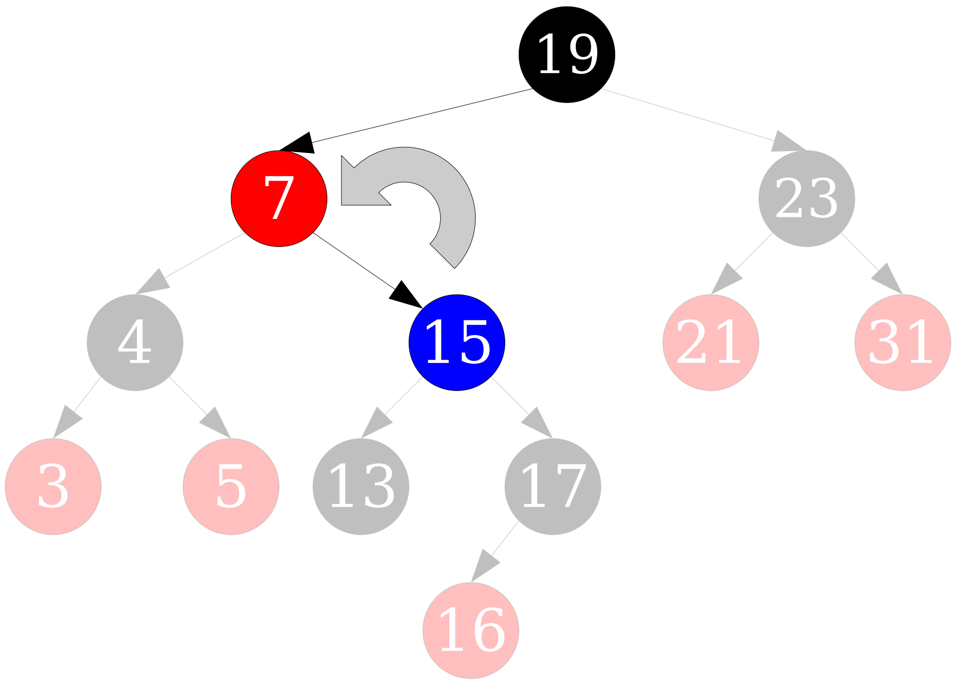

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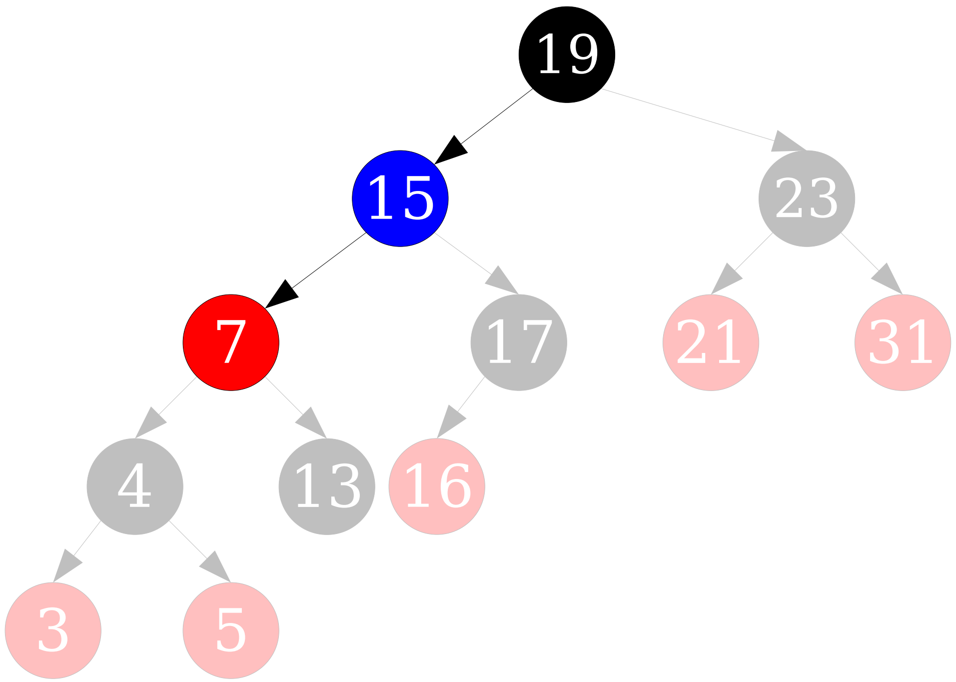
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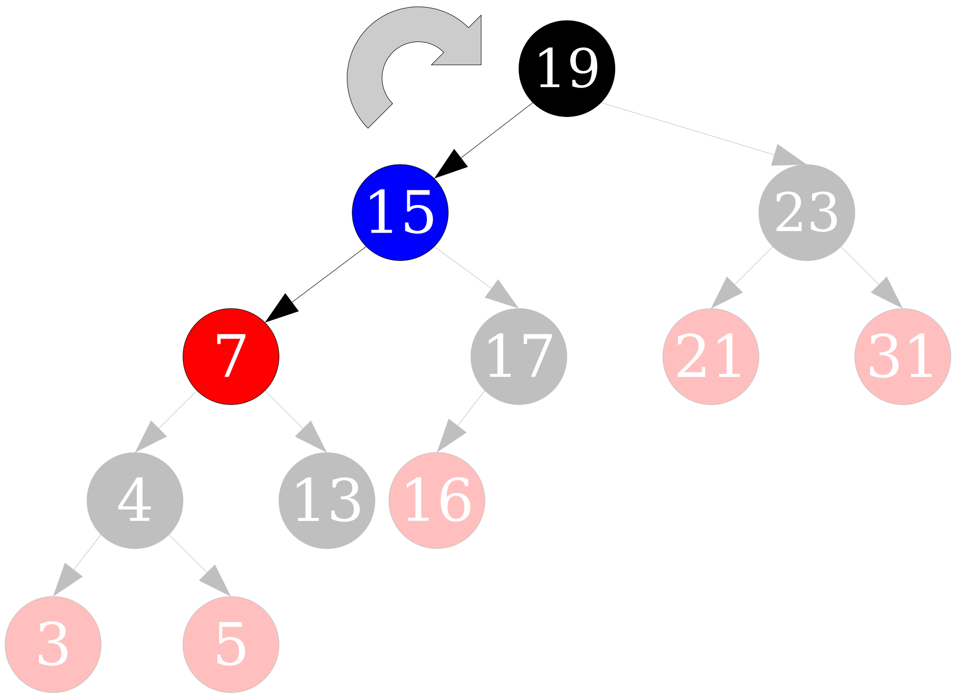
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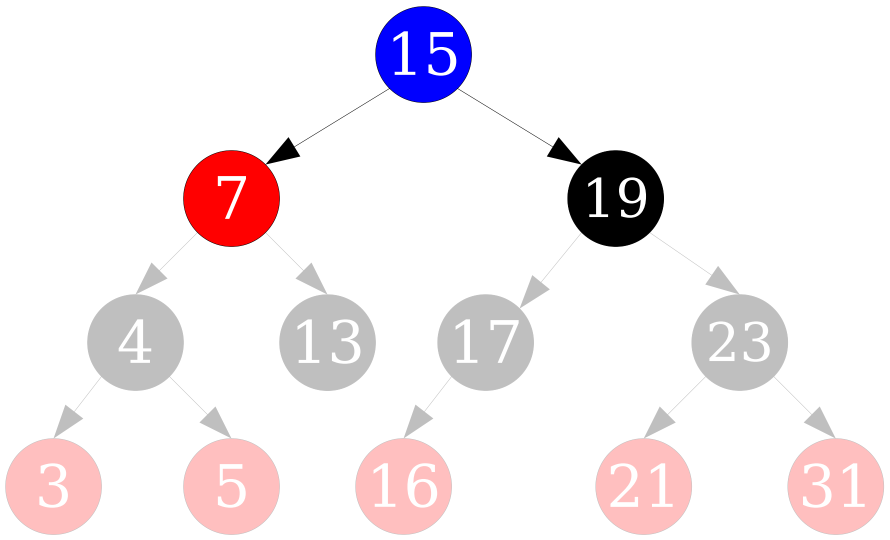
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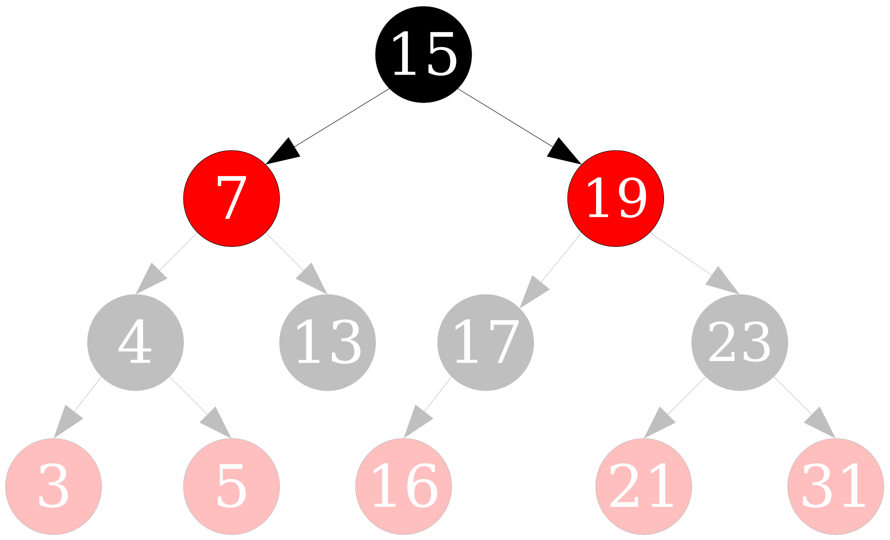
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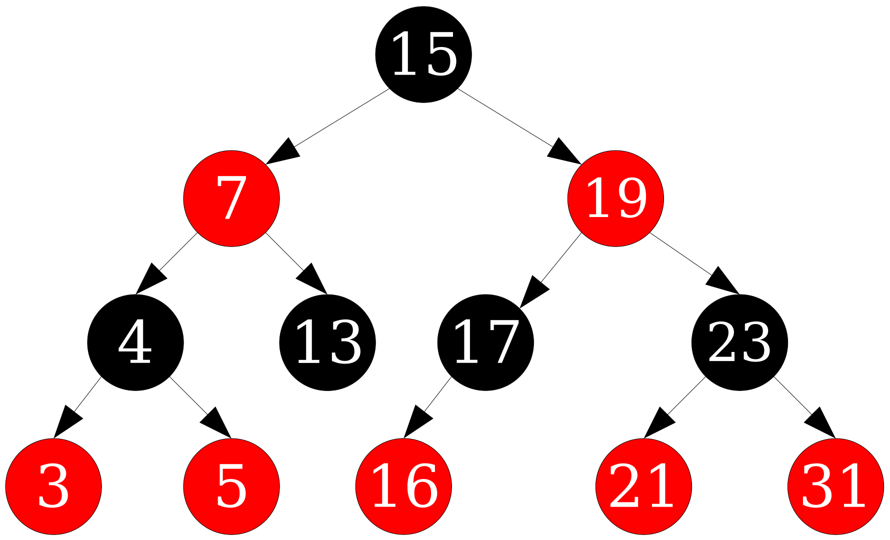
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Building Up Rules

- The complex rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.
- There are lots of cases to consider because there are many different ways you can insert into a red/black tree.
- ***Main point:*** Simulating the insertion of a key into a node takes time O(1) in all cases. Therefore, since 2-3-4 trees support O(log n) insertions, red/black trees support O(log n) insertions.
- The same is true of deletions.

A Few Useful Facts

- Every insertion into a red/black tree does at most two rotations.
- Every deletion from a red/black tree does at most three rotations.
- There may be O(log n) total color changes, but the “average” (amortized) number of color changes is O(1) per operation.

My Advice

- ***Do*** know how to do B-tree insertions and searches.
  - You can derive these easily if you remember to insert in leaves and split/kick up keys.
- ***Do*** remember the rules for red/black trees and B-trees.
  - These are useful for proving bounds and deriving results.
- ***Do*** remember the isometry between red/black trees and 2-3-4 trees.
  - They give a great intuition for all red/black tree operations.
- ***Don't*** memorize the red/black rotations and color flips.
  - This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. 😃

Time-Out for Announcements!

Participation Opt-Out Deadline

- As a reminder, the deadline to opt out of lecture participation and shift the weight to your final exam is this Friday at 11:59PM.
- You can do so by filling out the form linked from the EdStem Q&A forum.

Problem Set 3

- Problem Set 2 was due today at 1:00PM.
  - Need more time? Feel free to use one or two late days to extend the deadline by 24 or 48 hours, respectively.
- Problem Set 3 (***Hashing and Sketching***) goes out today at 1:00PM. It comes due ***Tuesday, May 5*** at ***1:00PM***.
  - Get a feel for how 2-independent hashing works.
  - Design your own cardinality estimator and learn to work with concentration inequalities.
  - Implement HyperLogLog and a modern variant and see just how good it is.
- Ping us on Ed or stop by office hours if you have questions!

Back to CS166!

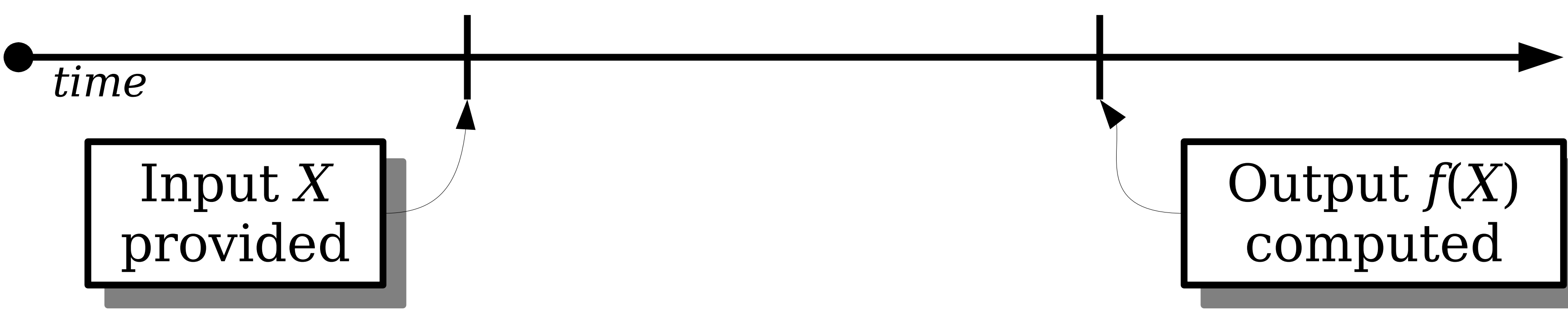
Dynamic Problems

Classical Algorithms

- The "classical" algorithms model goes something like this:

Given some input X, compute some interesting function f(X).

- The input X is provided up front, and only a single answer is produced.

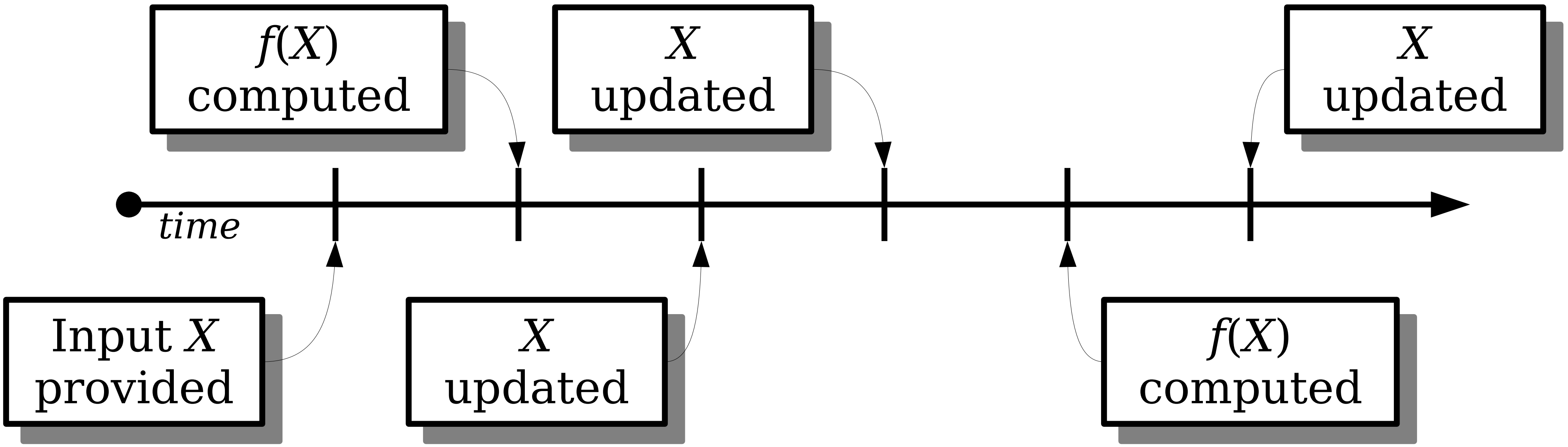


Dynamic Problems

- ***Dynamic versions*** of problems are framed like this:

  Given an input X that can change in particular ways, maintain X while being able to compute f(X) efficiently at any point in time.

- These problems are typically harder to solve efficiently than the “classical” static versions.



Dynamic Selection

- The ***selection*** problem is the following:

Given a list of distinct values and a number k, return the kth-smallest value.

- In the static case, where the data are fixed in advance and k is known, we can solve this in time O(n) using quickselect or the median-of-medians algorithm.
- ***Goal:*** Solve this problem efficiently when the data set is changing – that is, the underlying set of elements can have insertions and deletions intermixed with queries.

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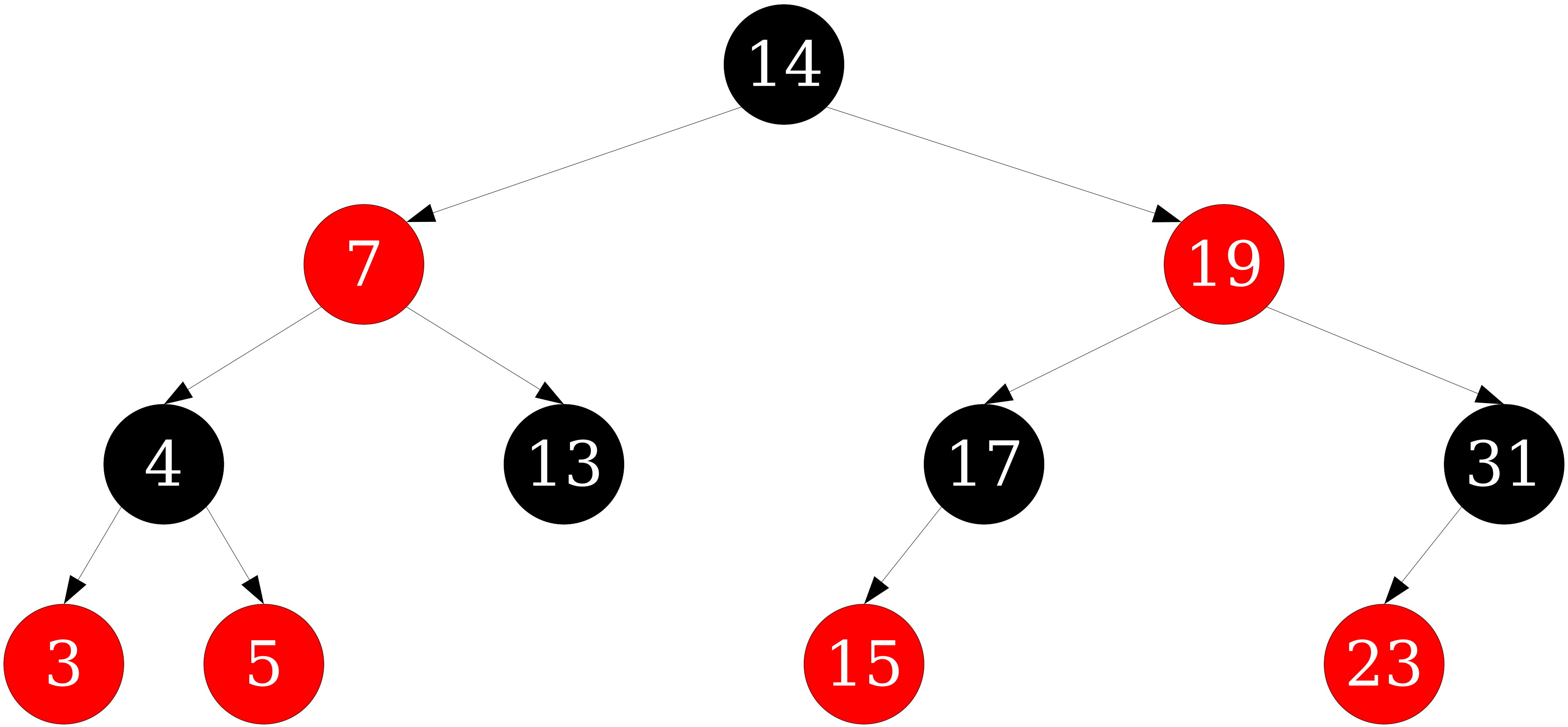

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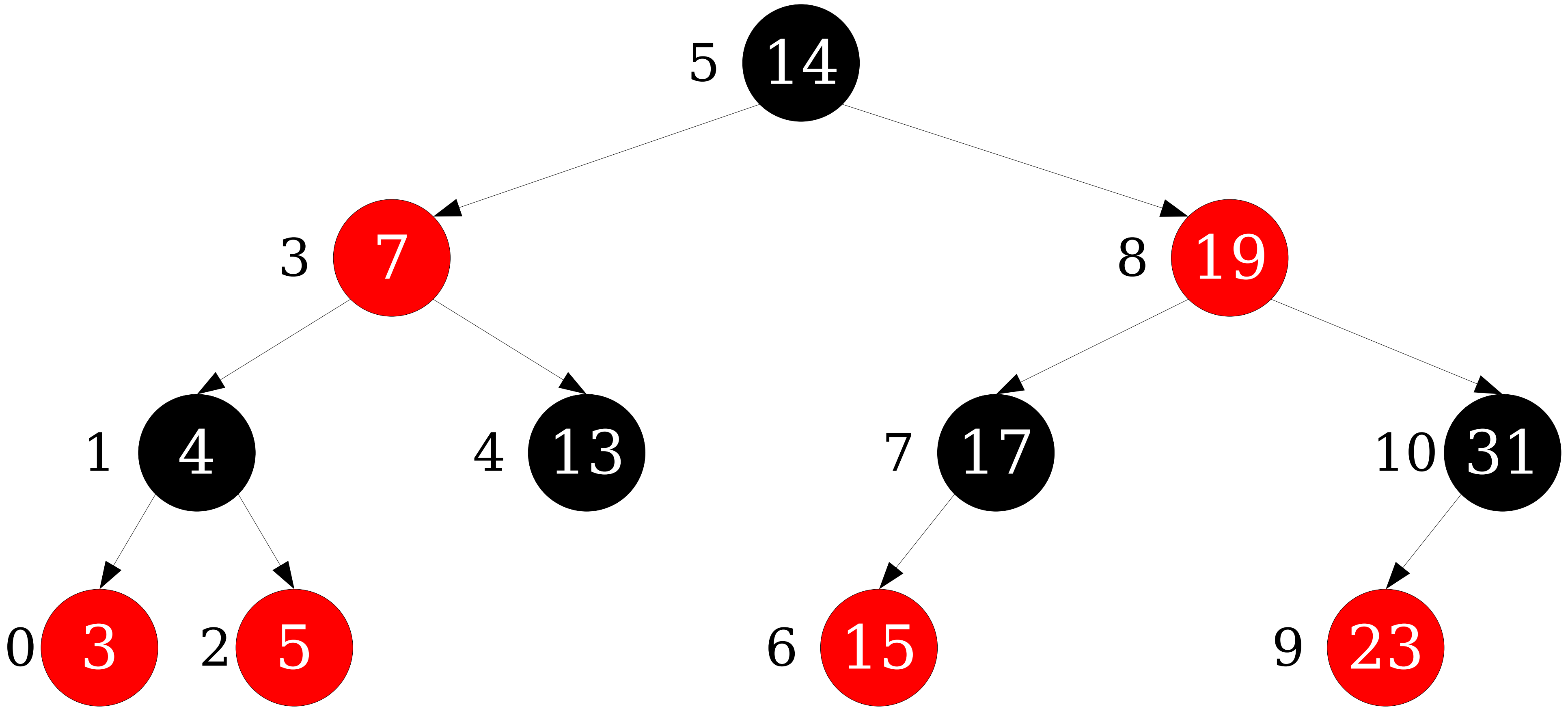

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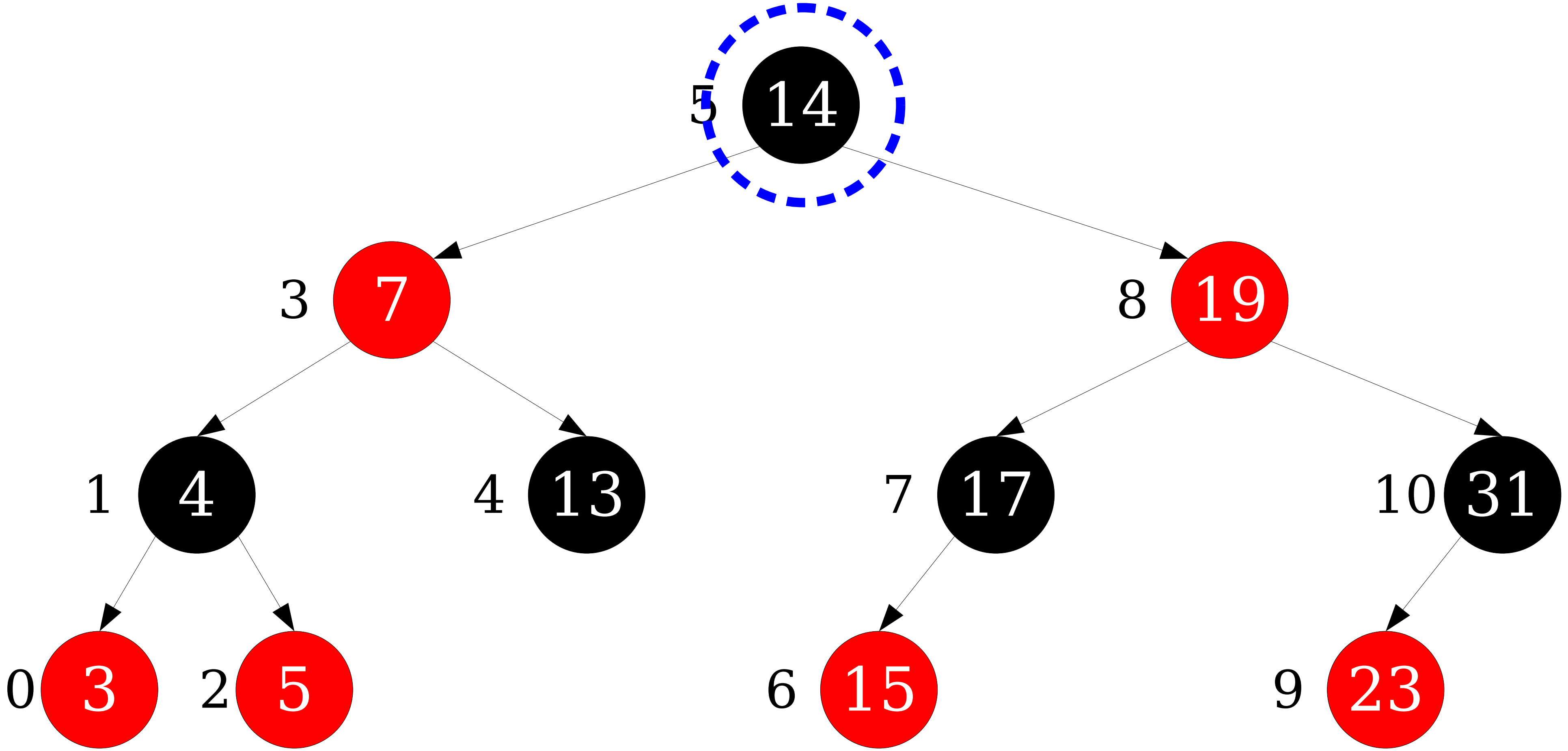

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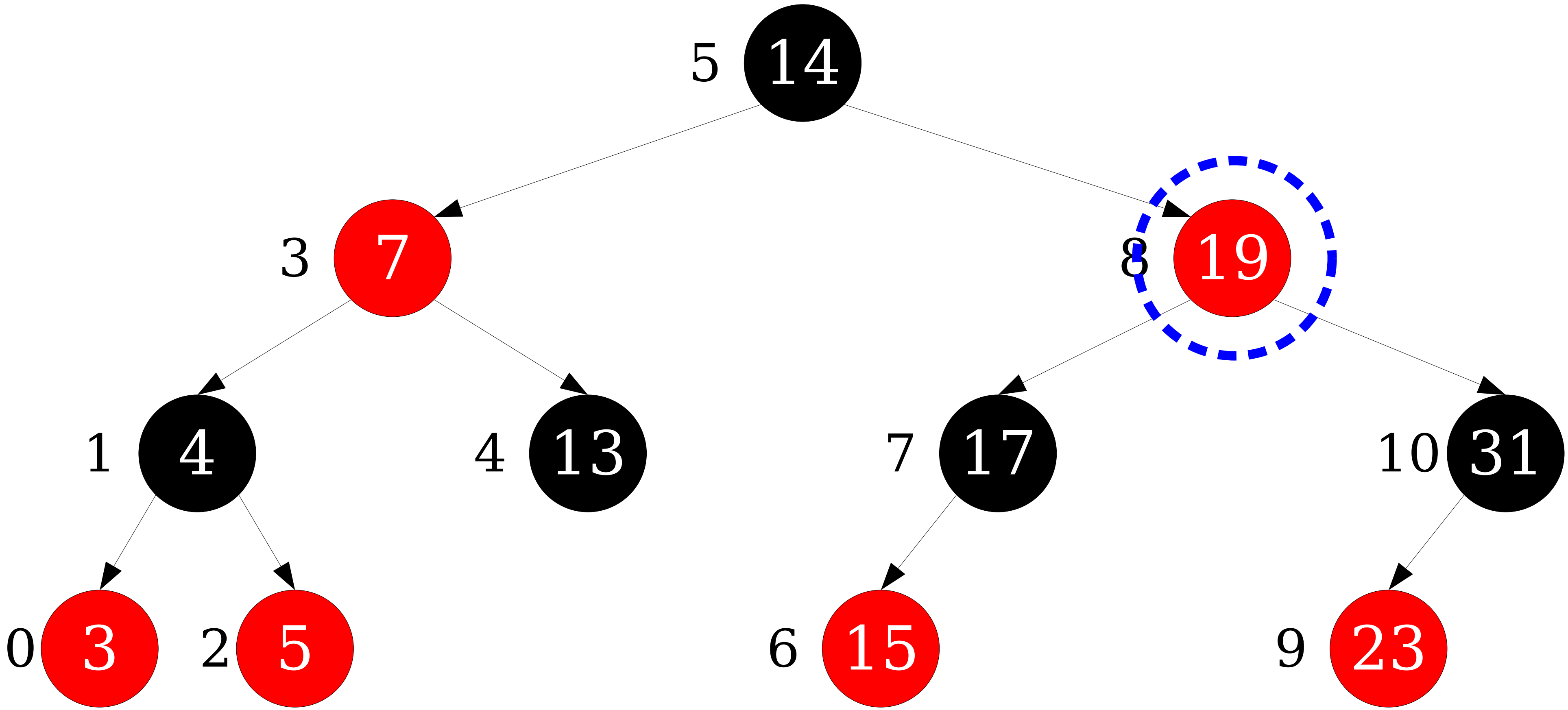

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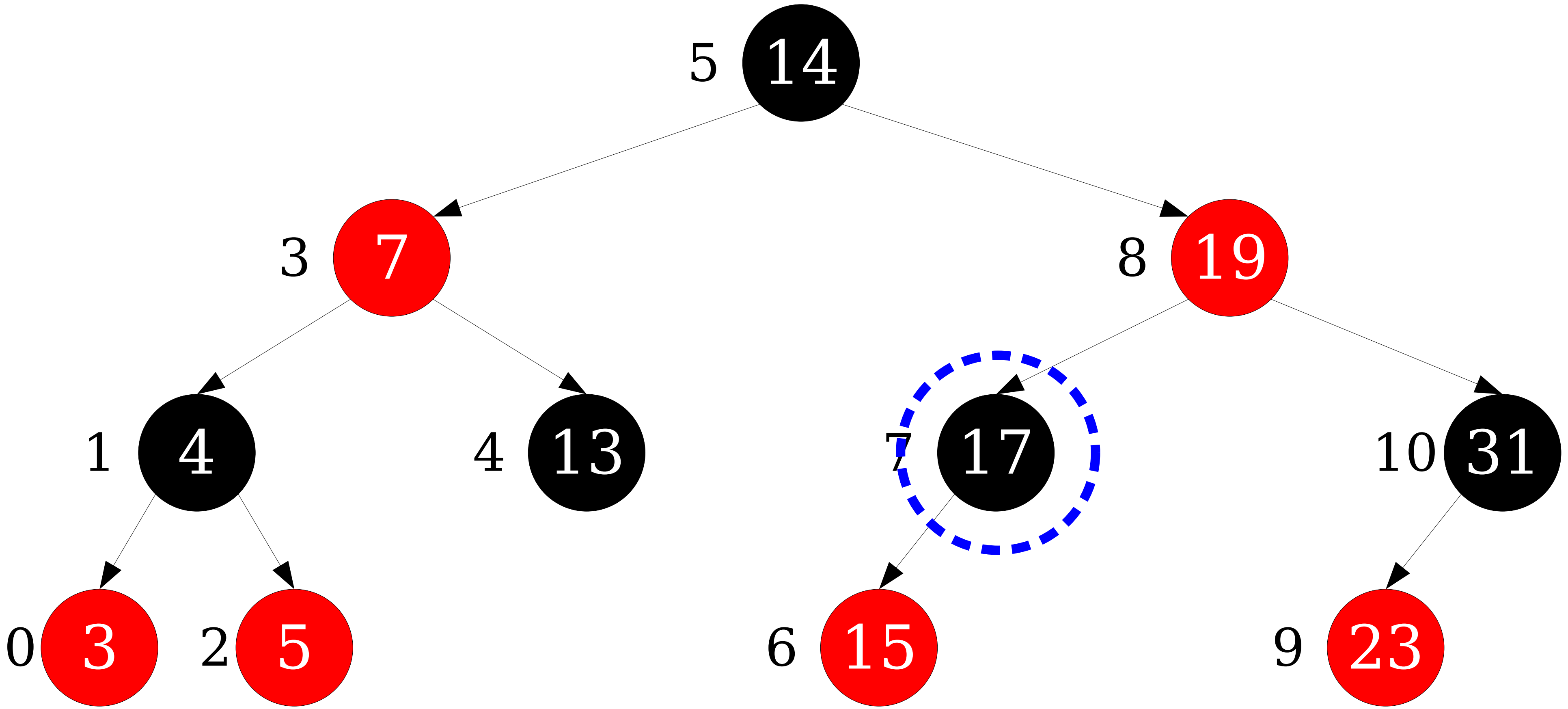

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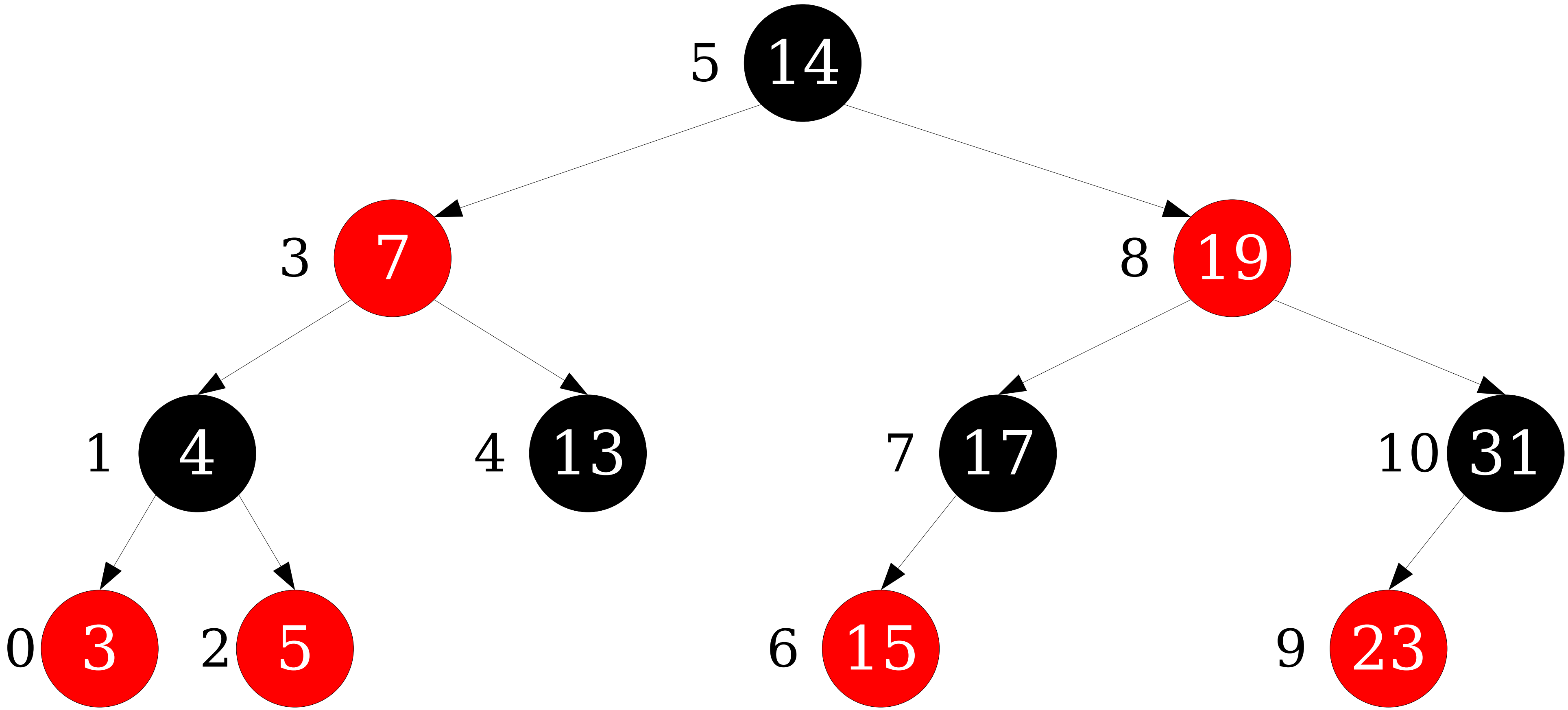

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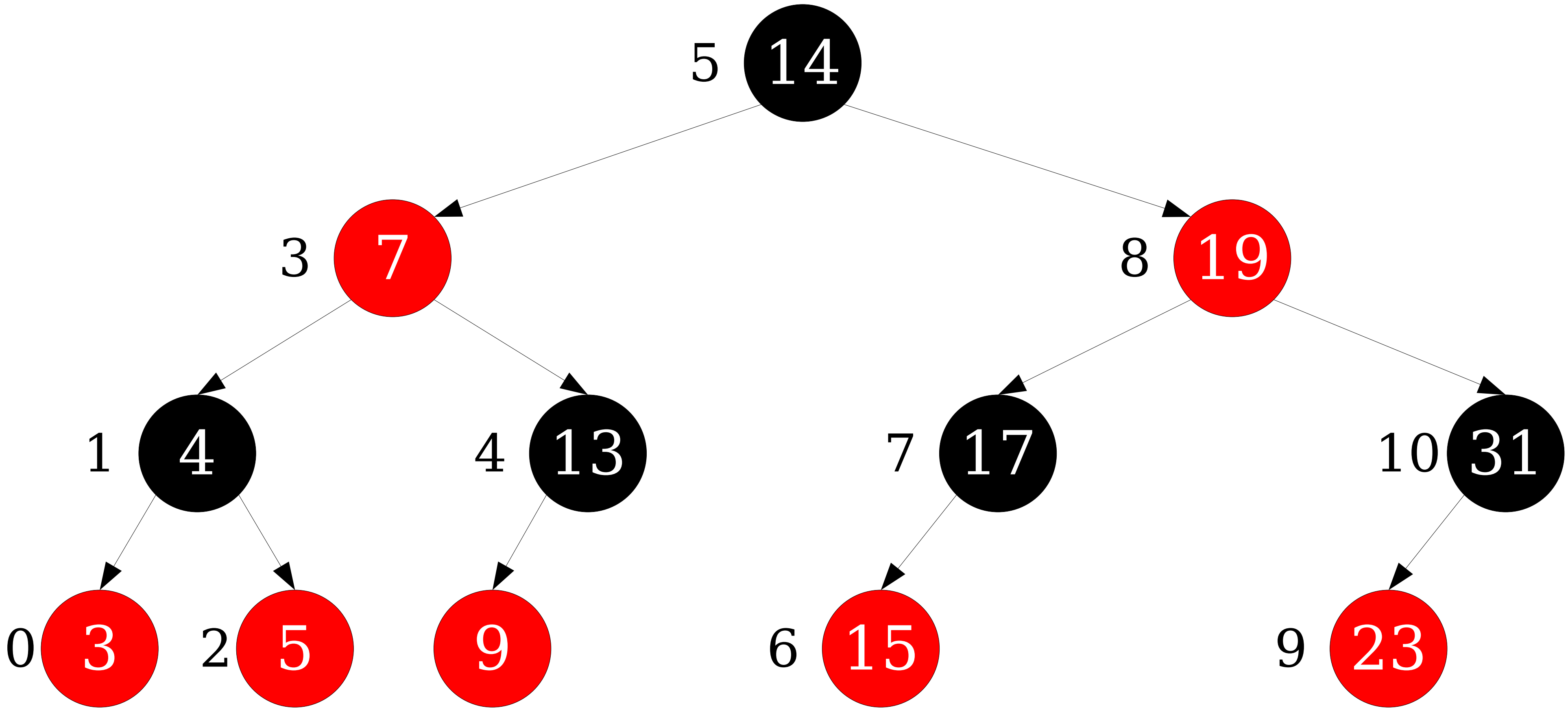

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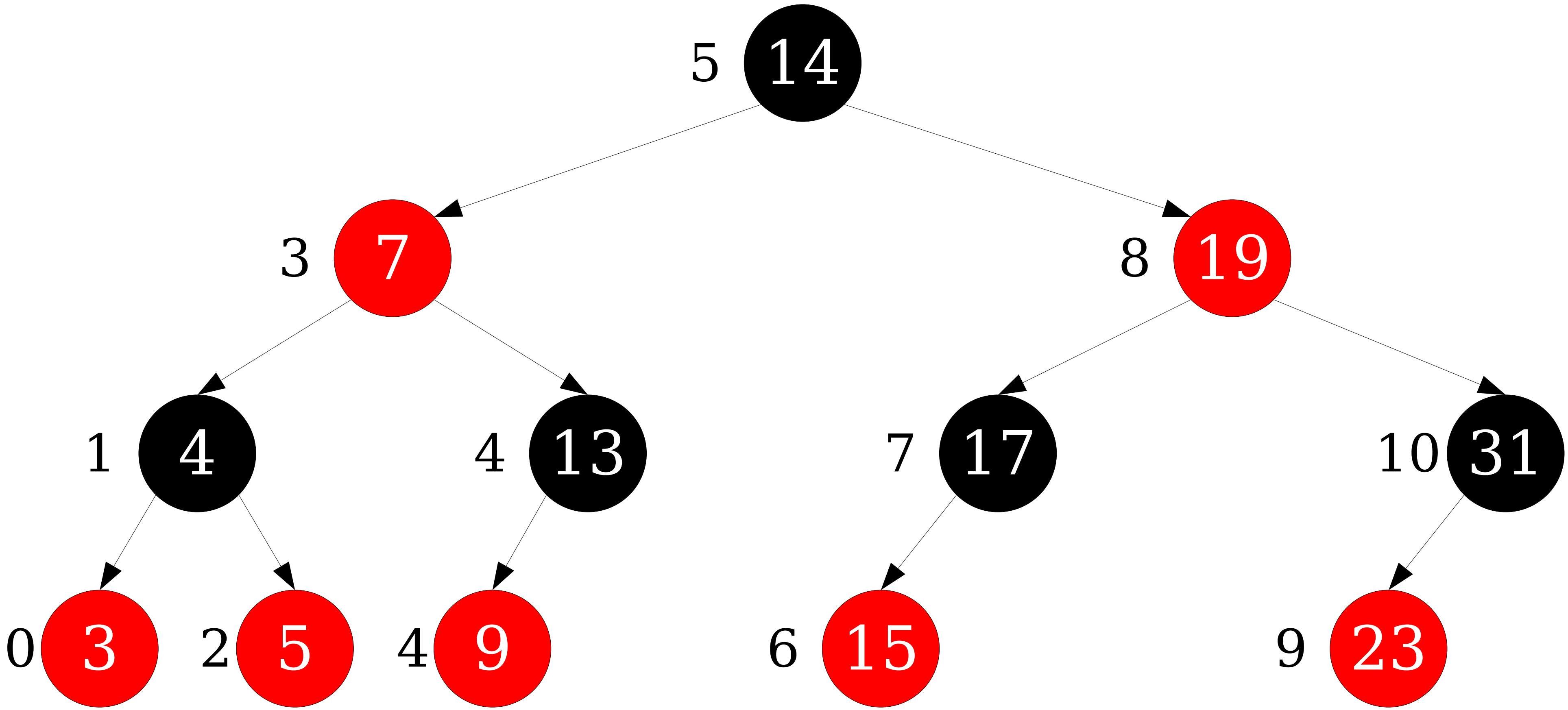

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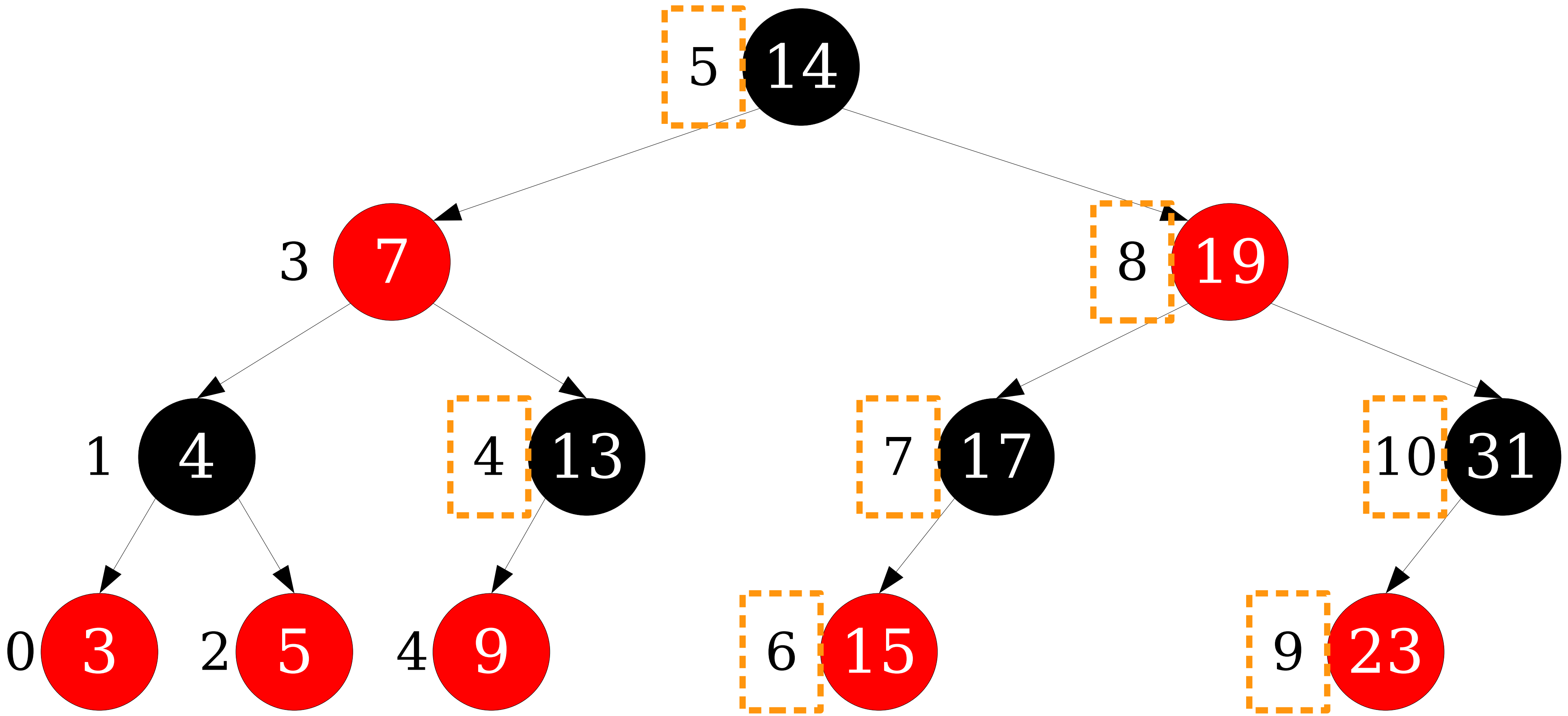

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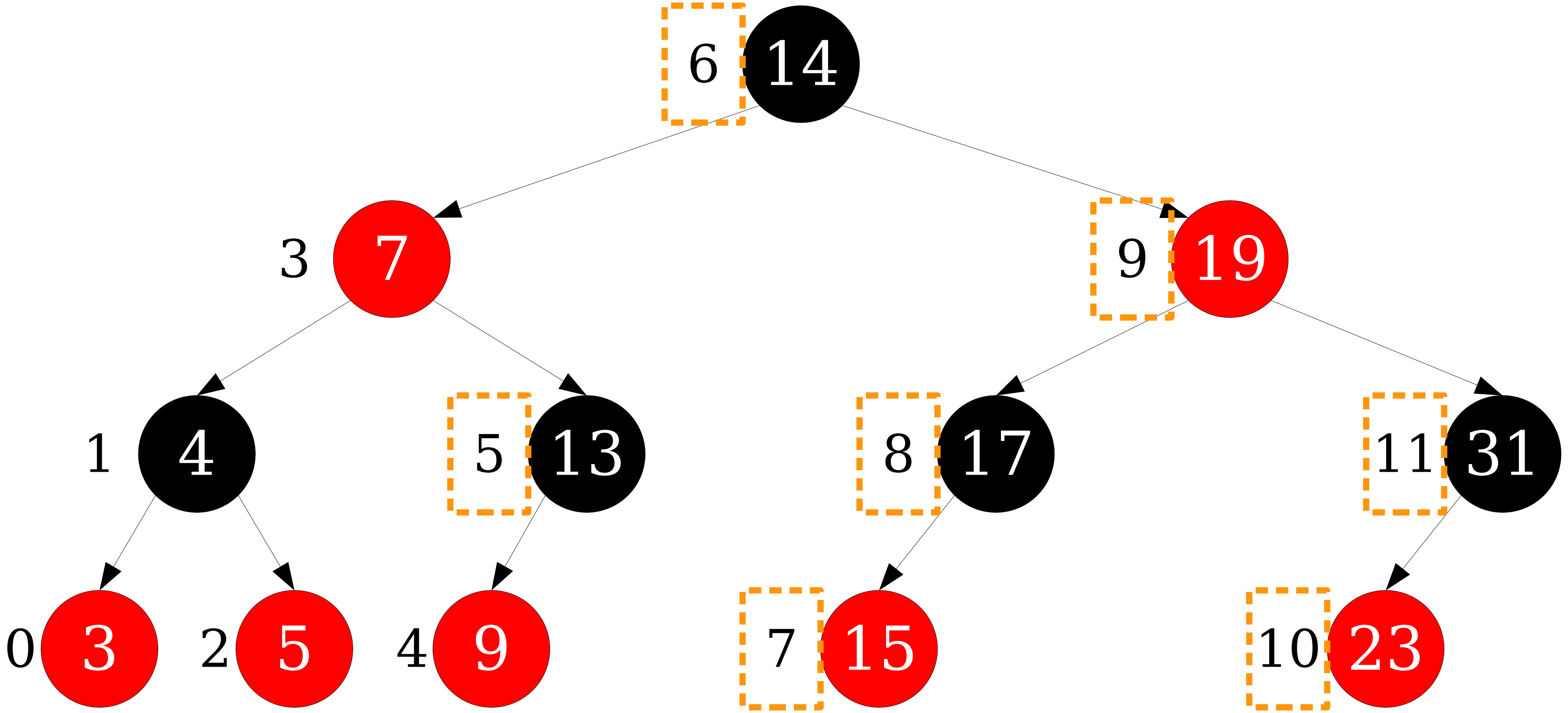

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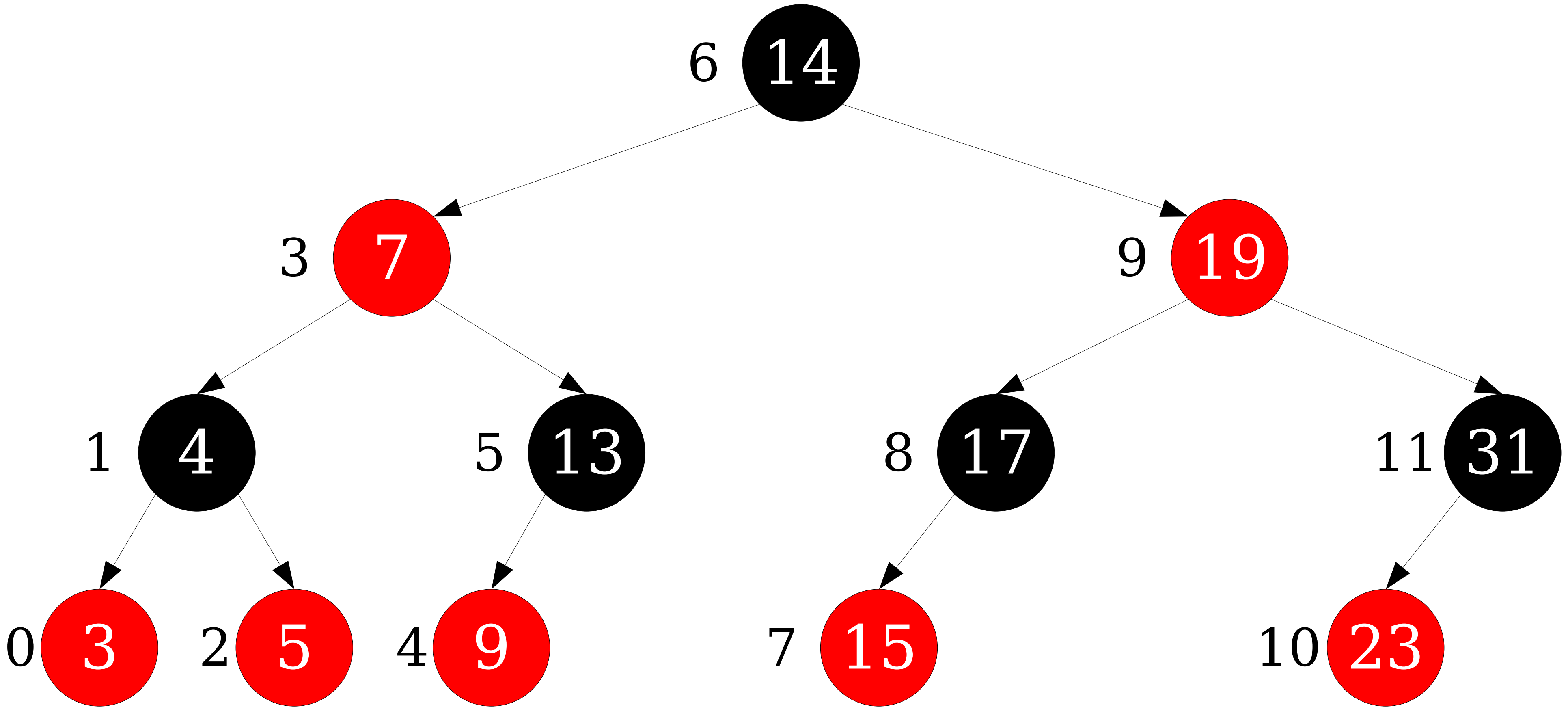

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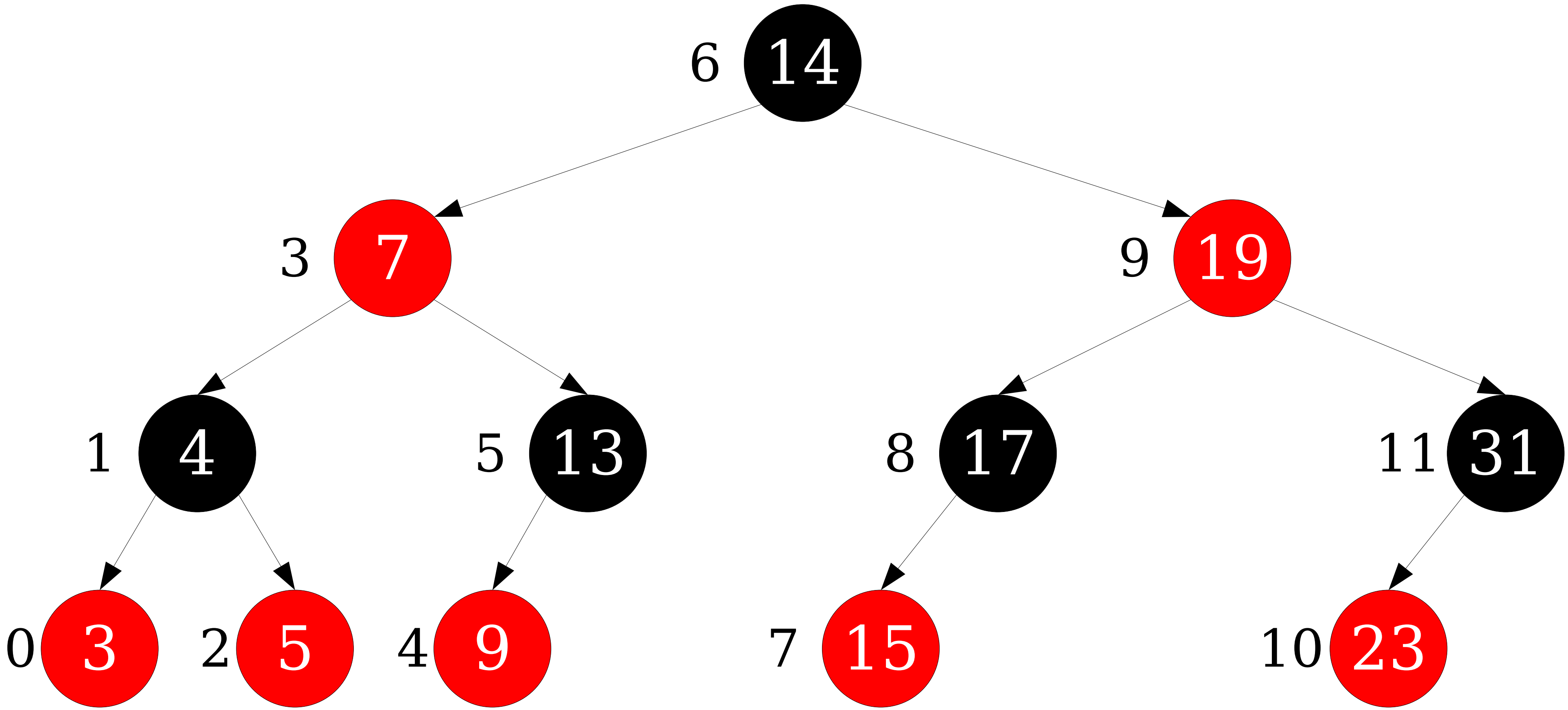

Problem: After inserting a new value, we may have to update $\Theta(n)$ values.

This is inherent in this solution route. These numbers track *global* properties of the tree.

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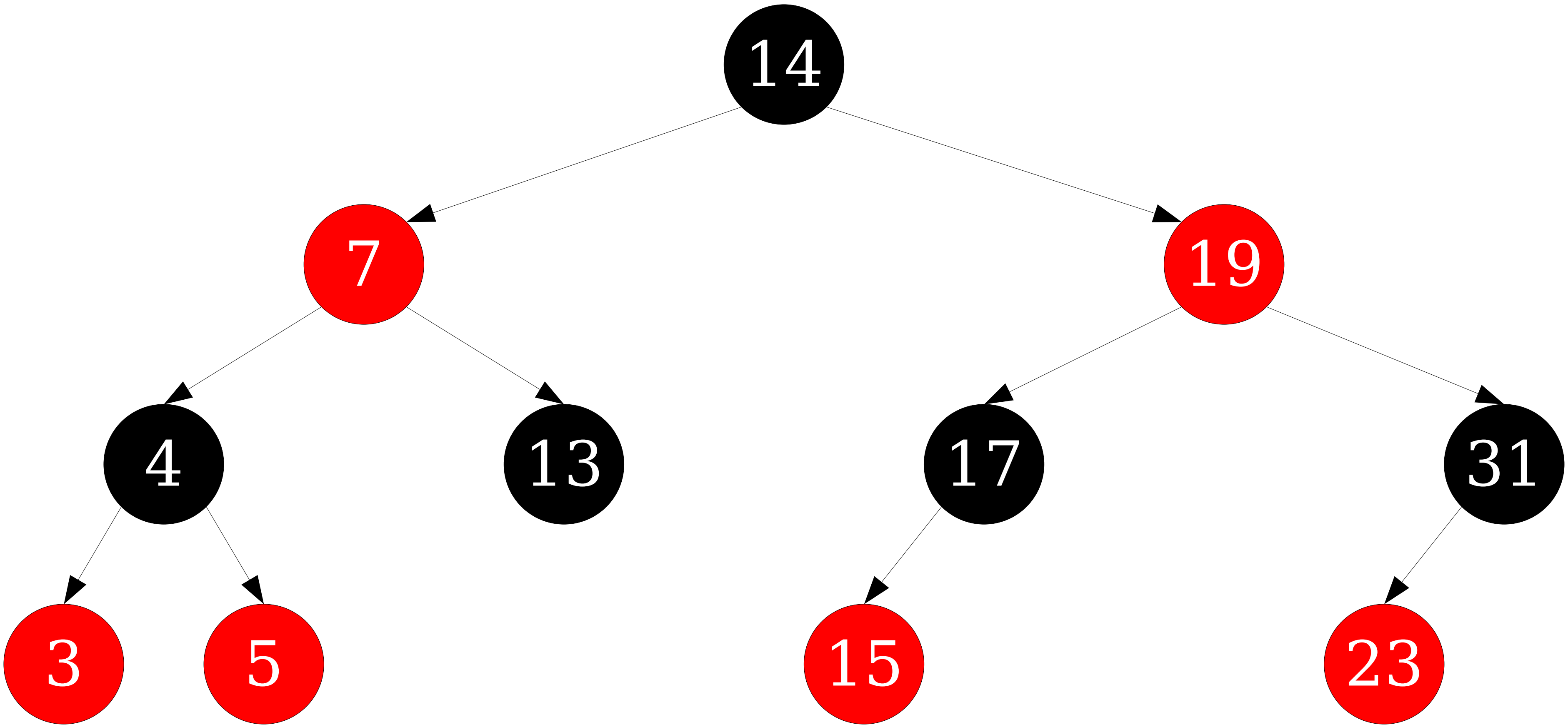

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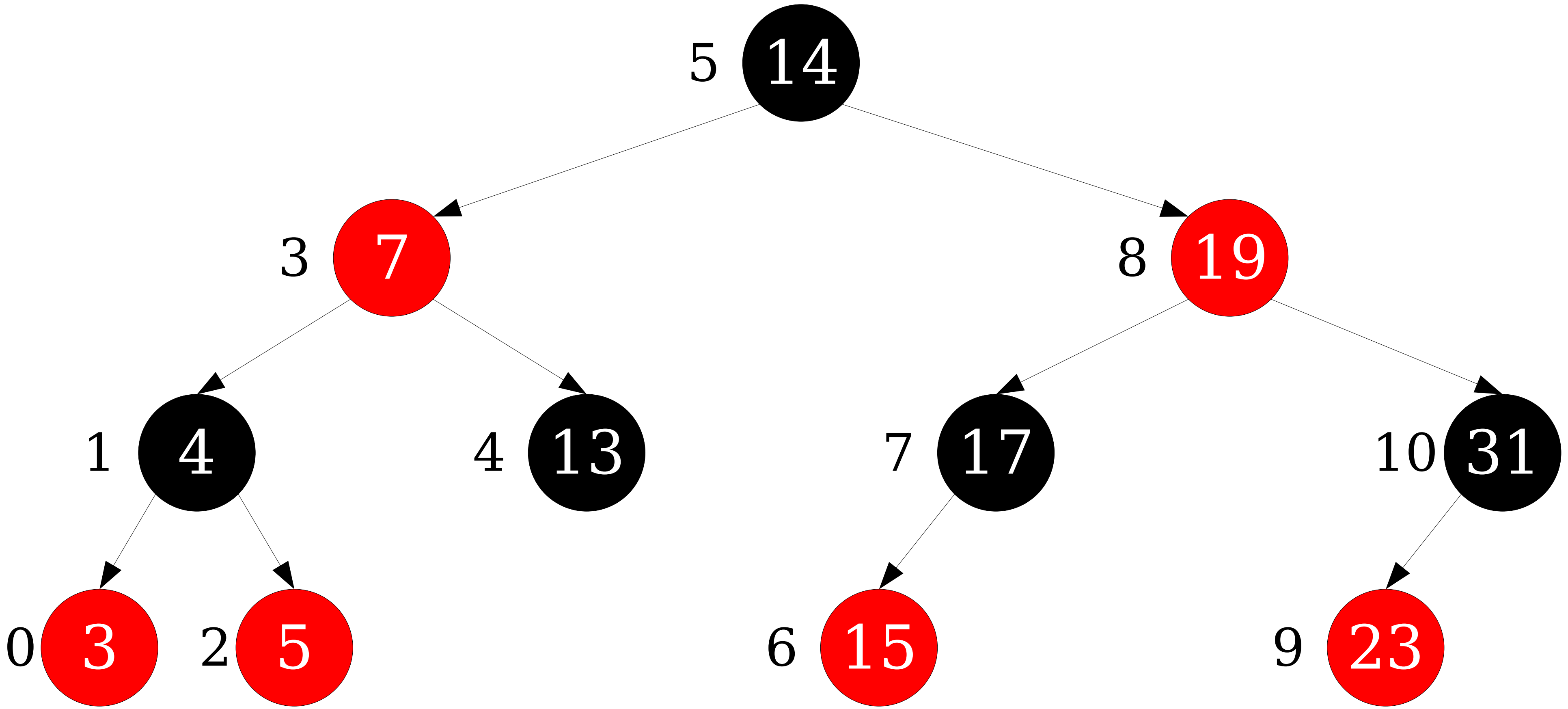

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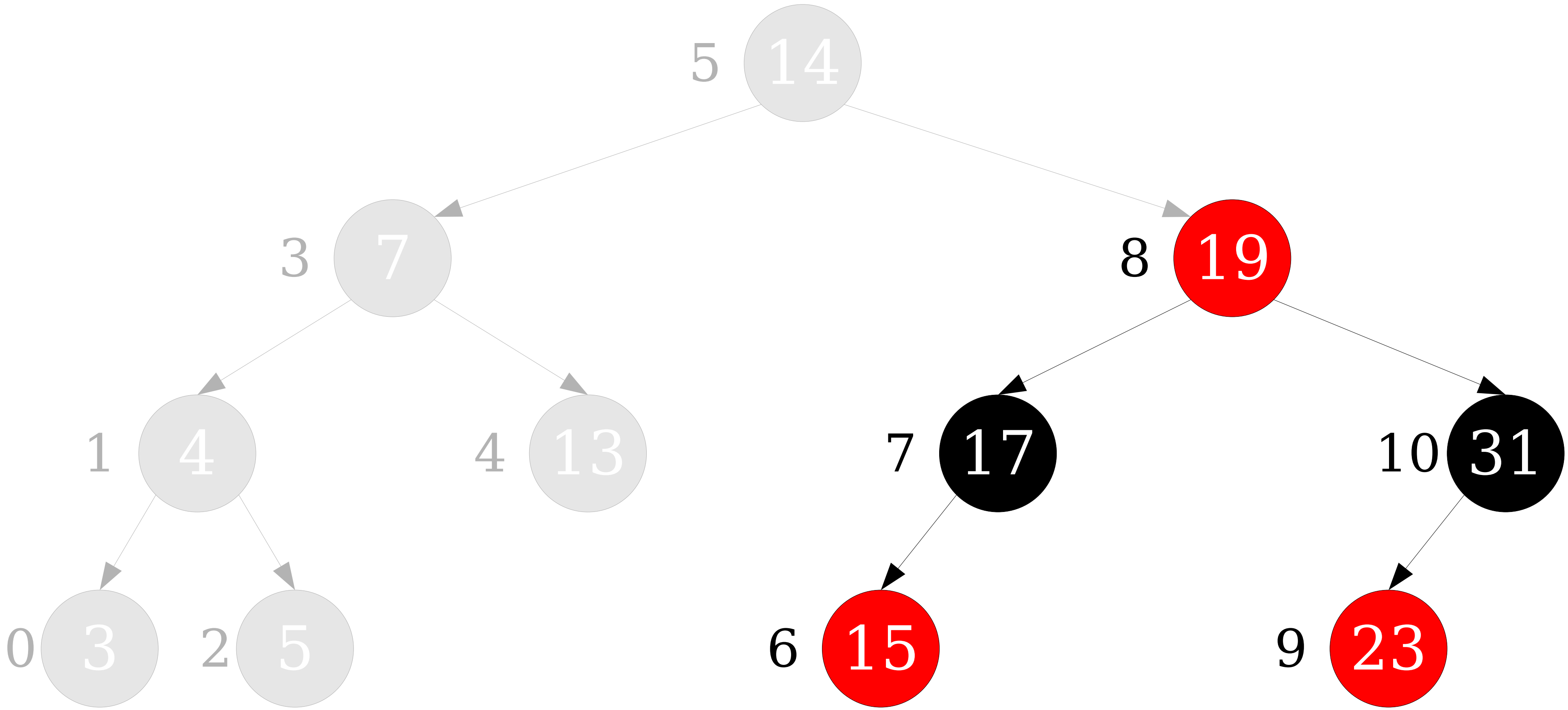

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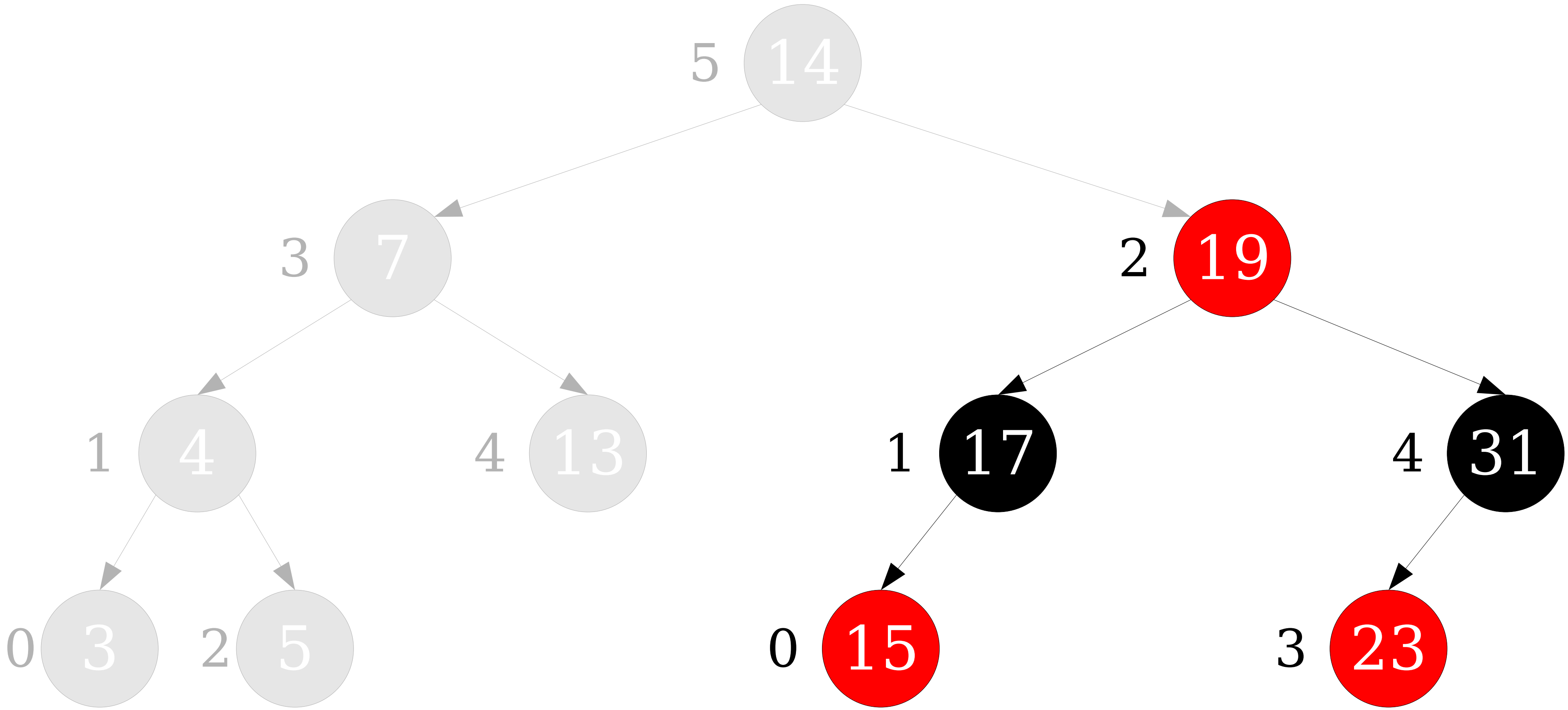

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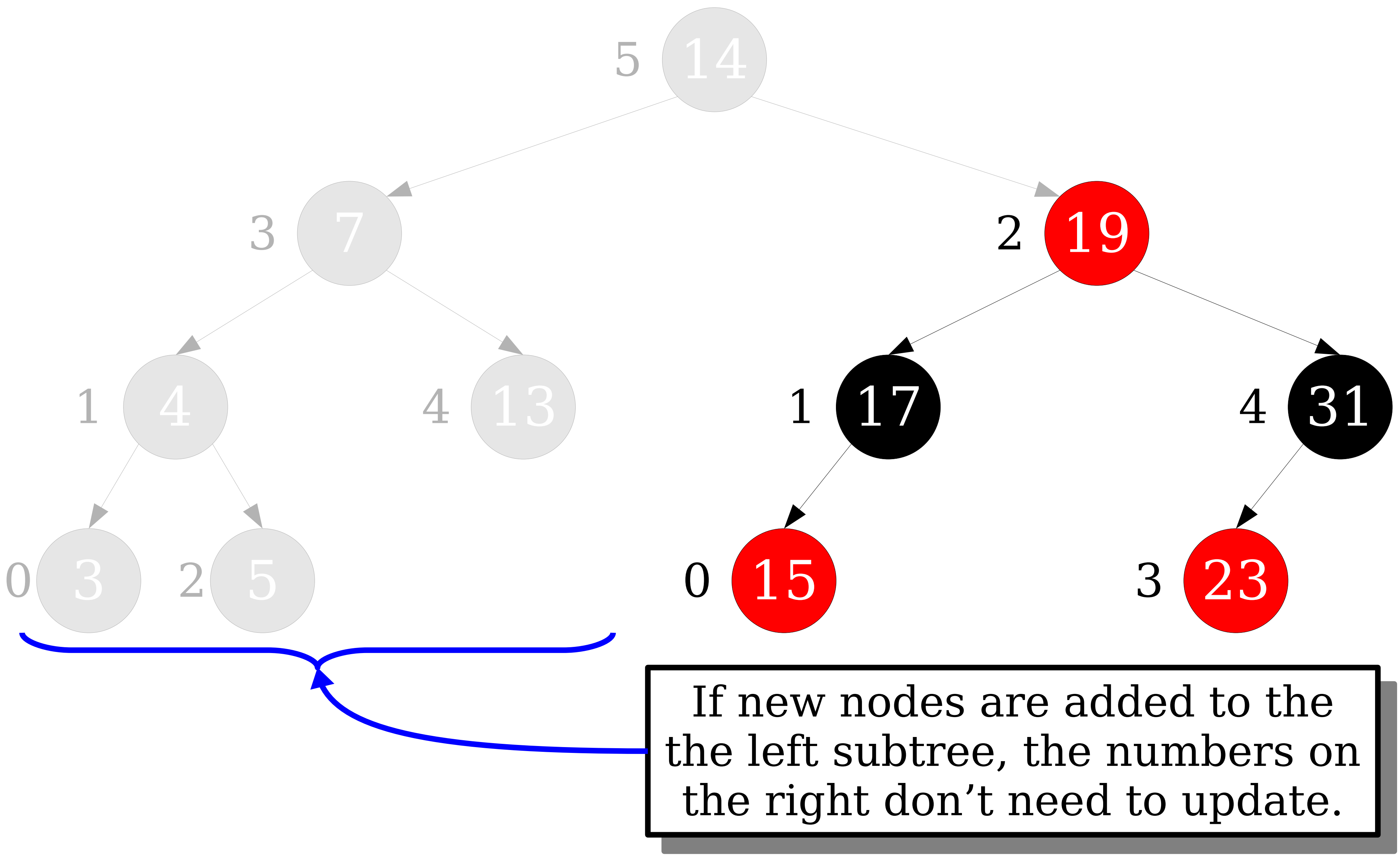

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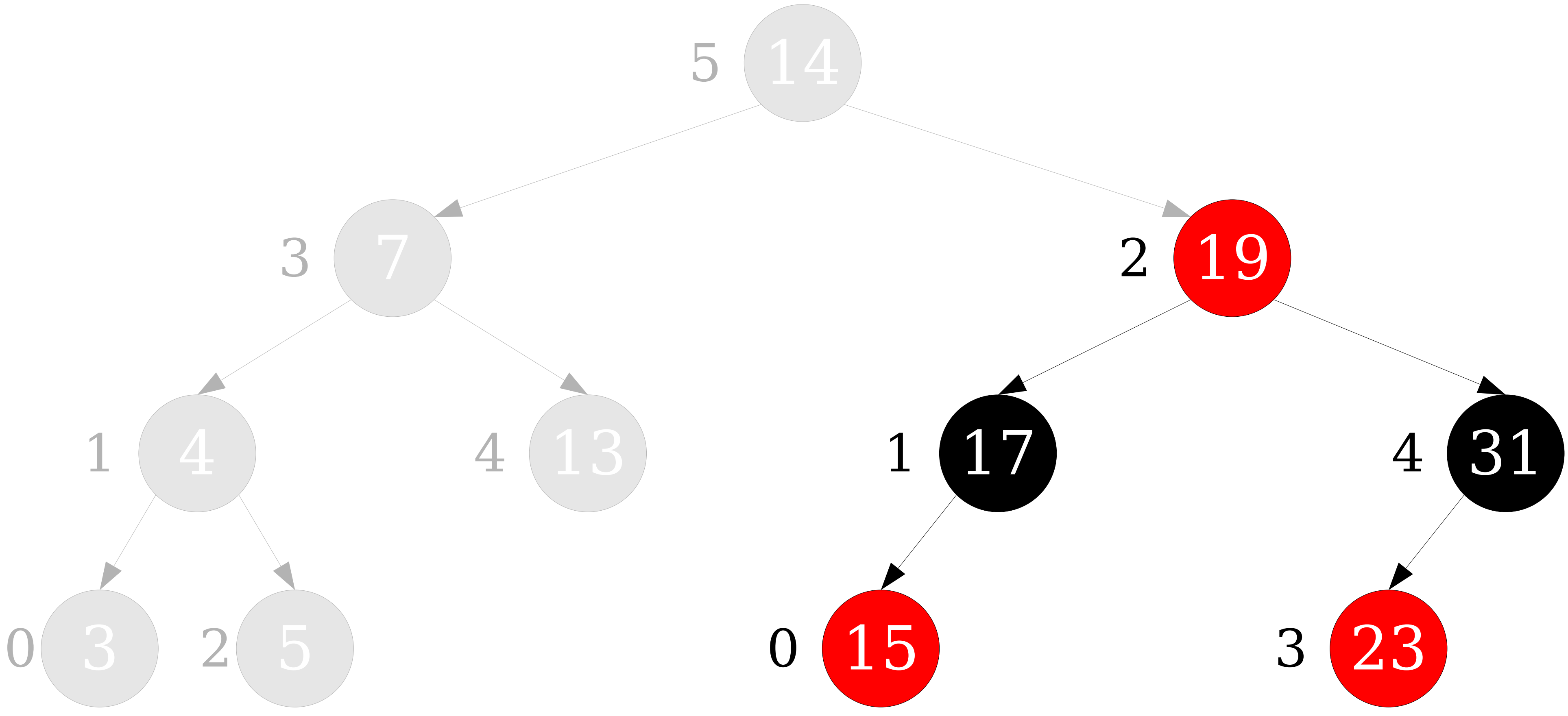

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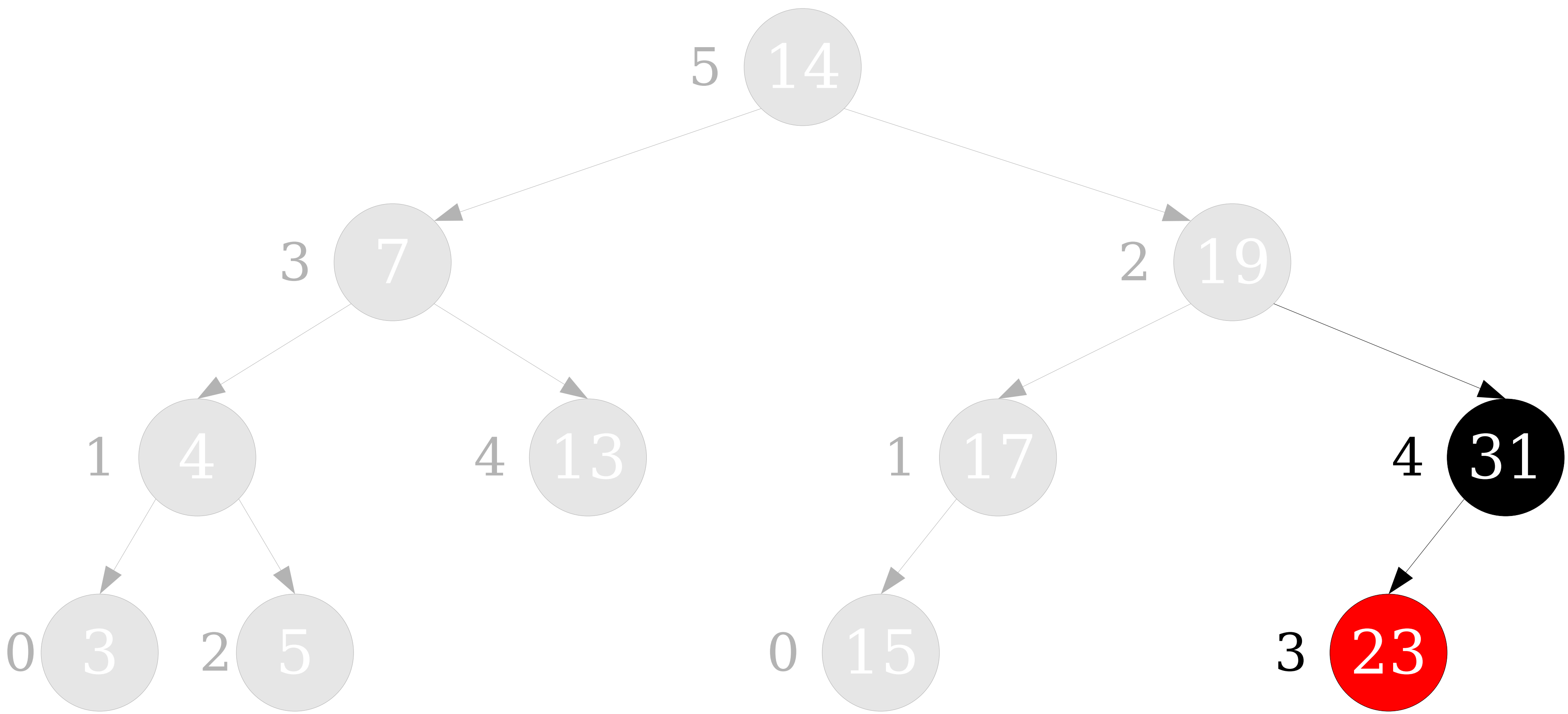

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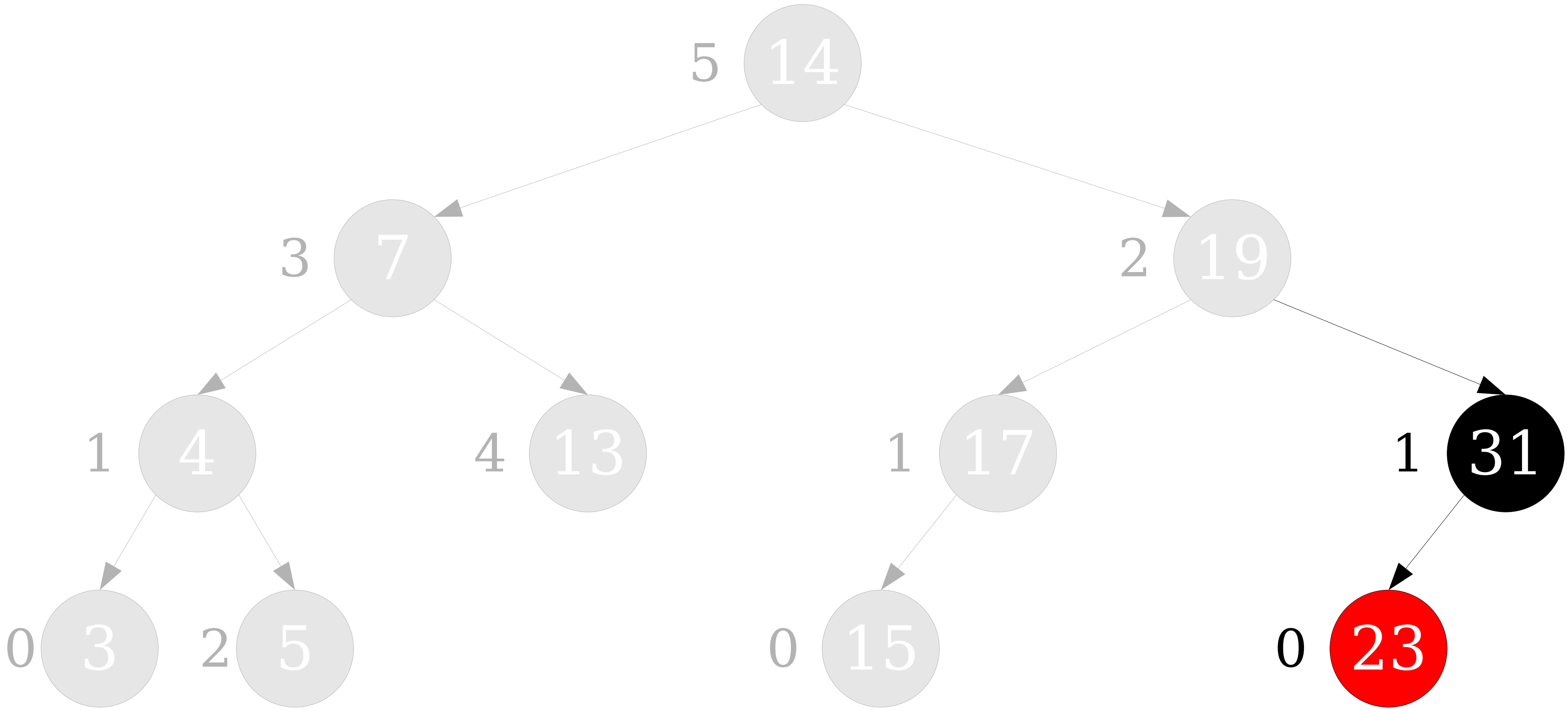

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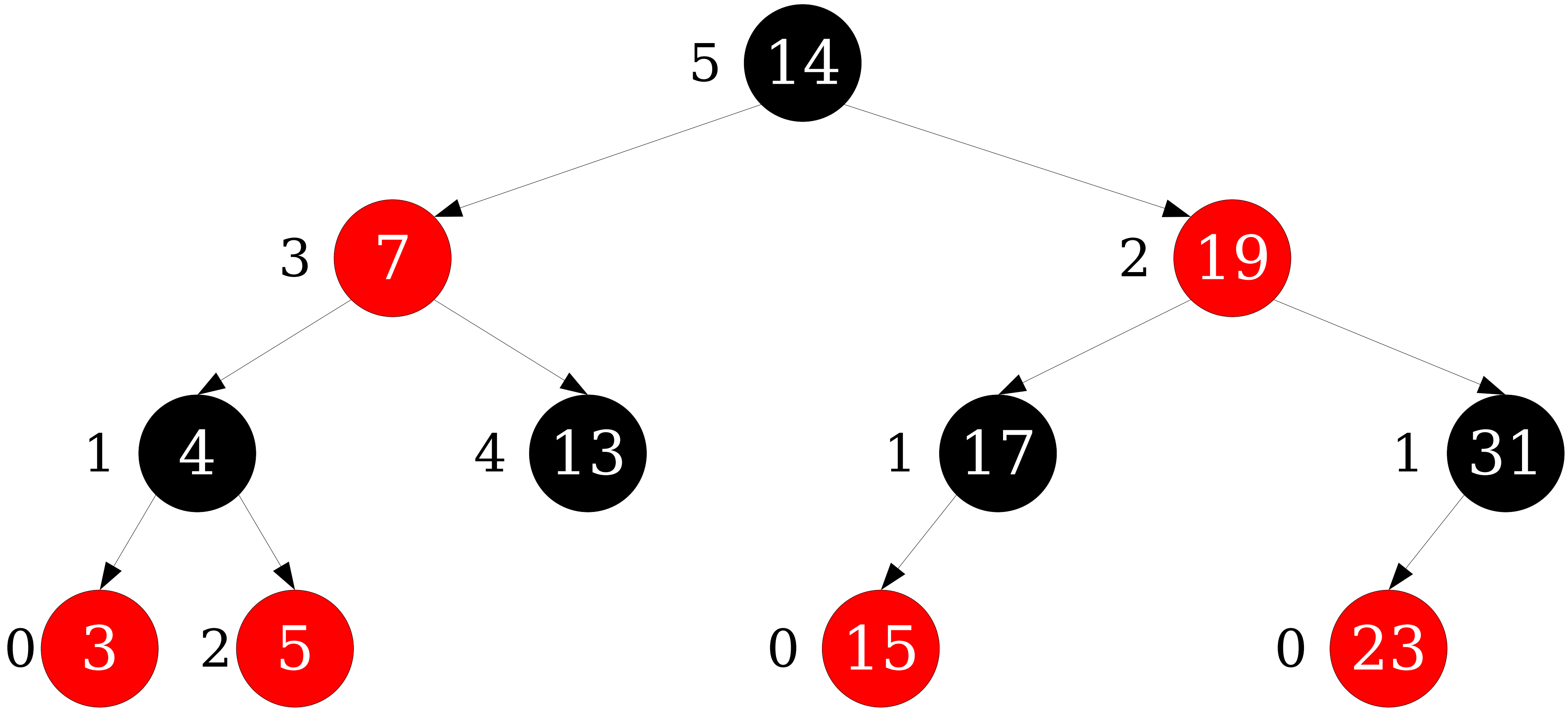

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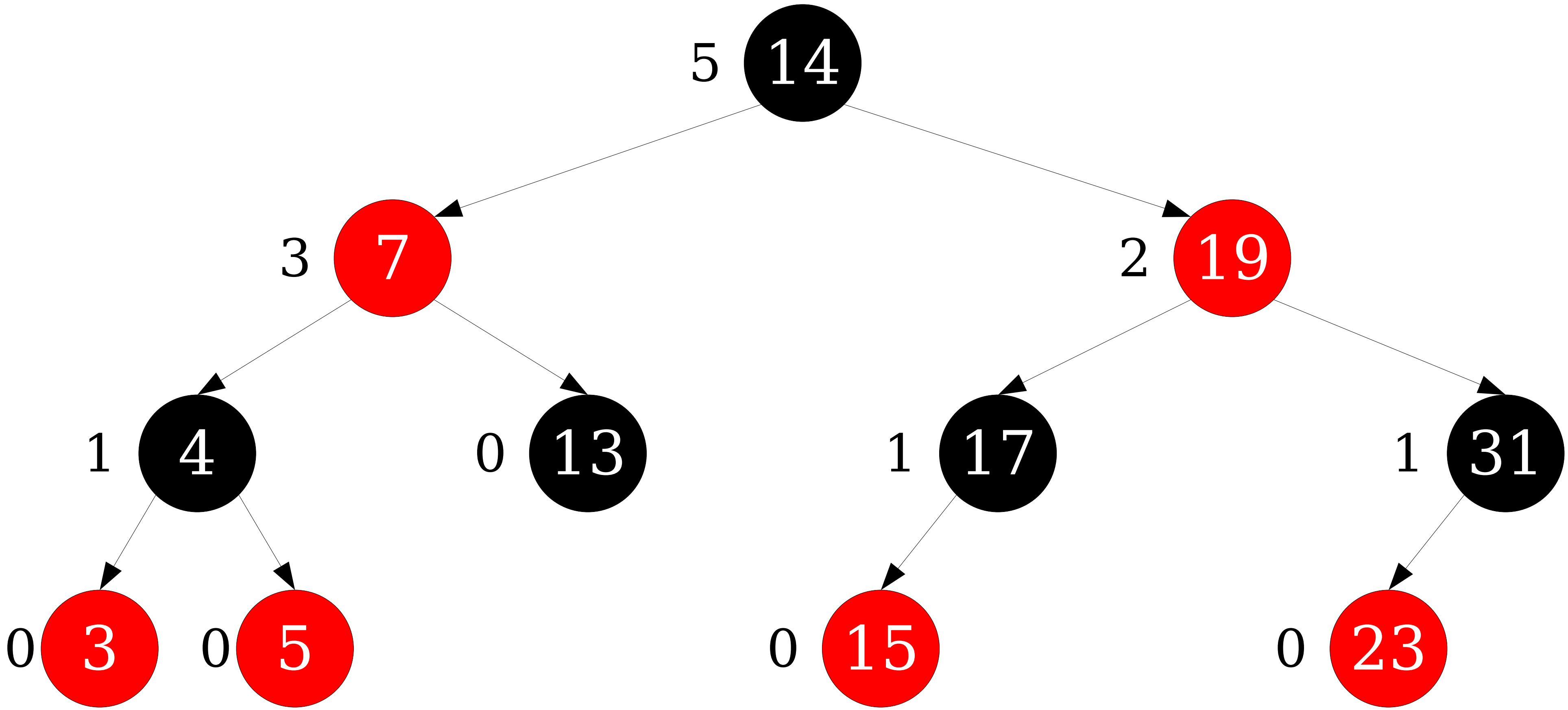

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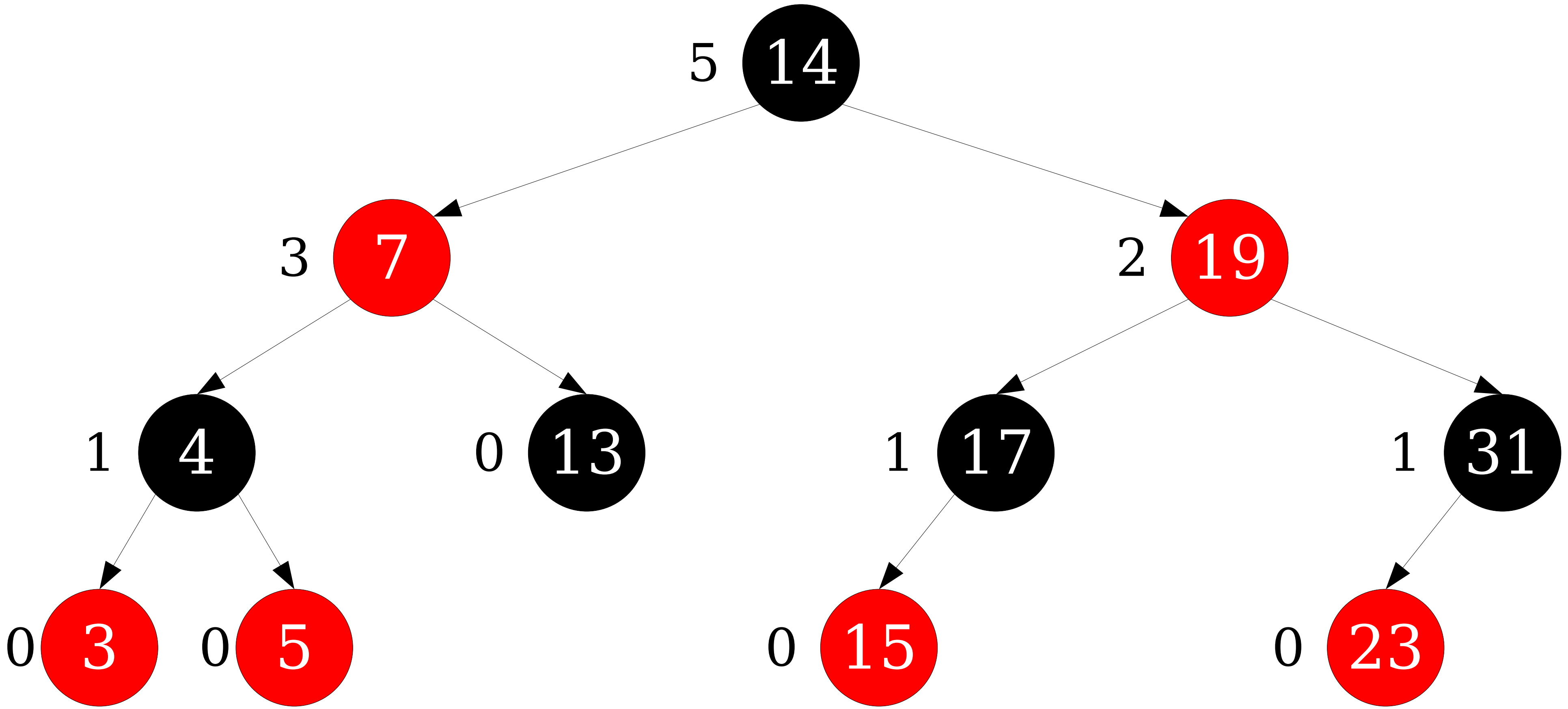

Perspective 1: Tag each key with the its rank in the subtree rooted at its node.

Perspective 2: Annotate each key with the number of keys in its left subtree.

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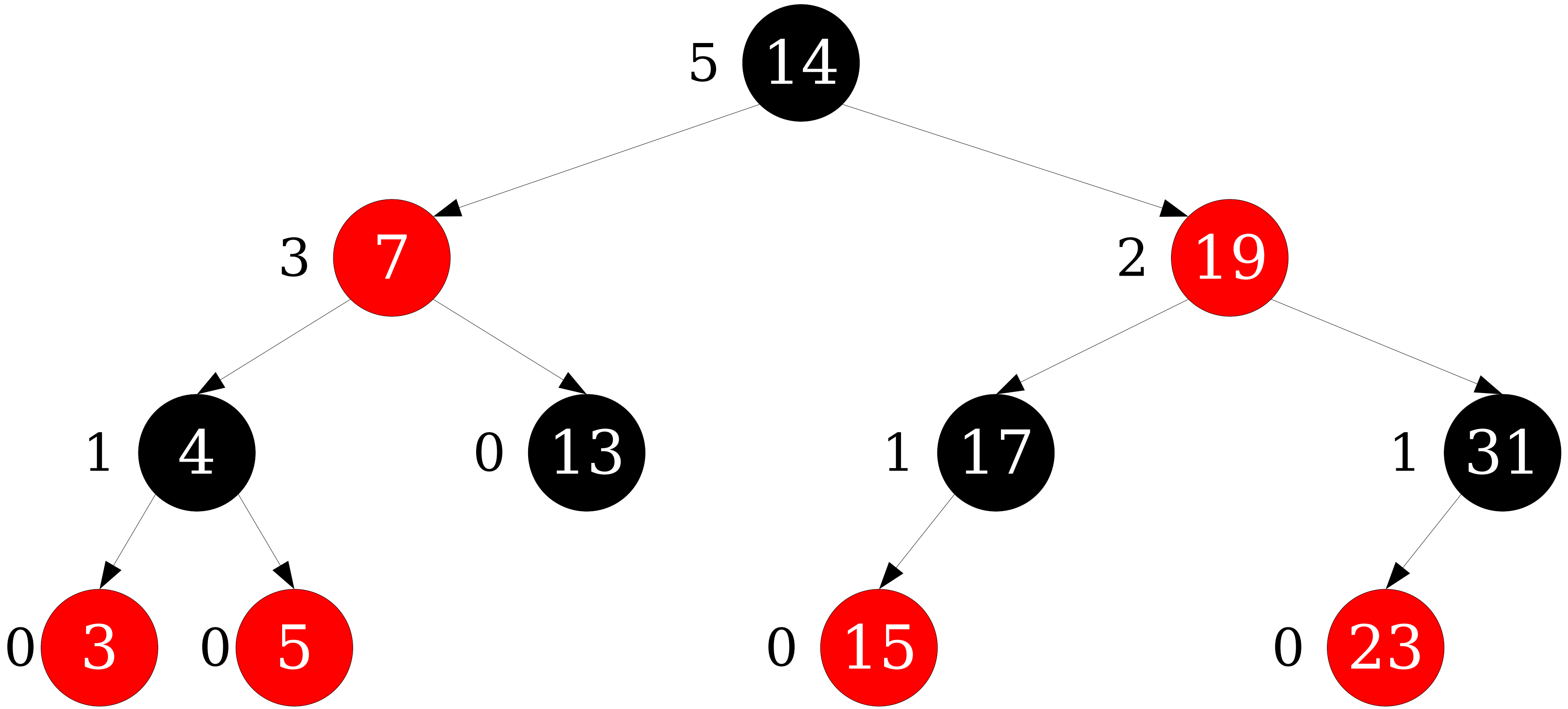

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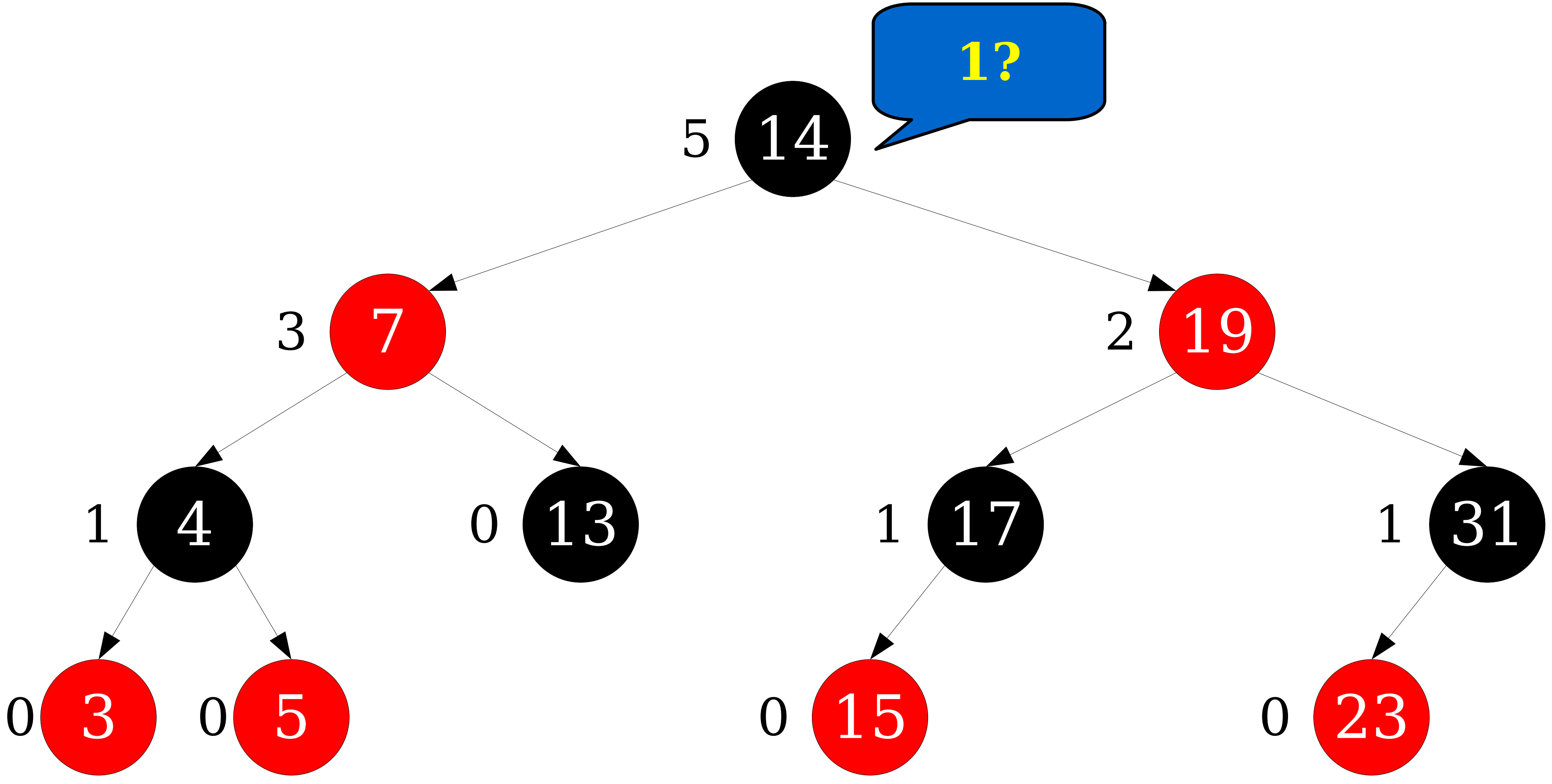

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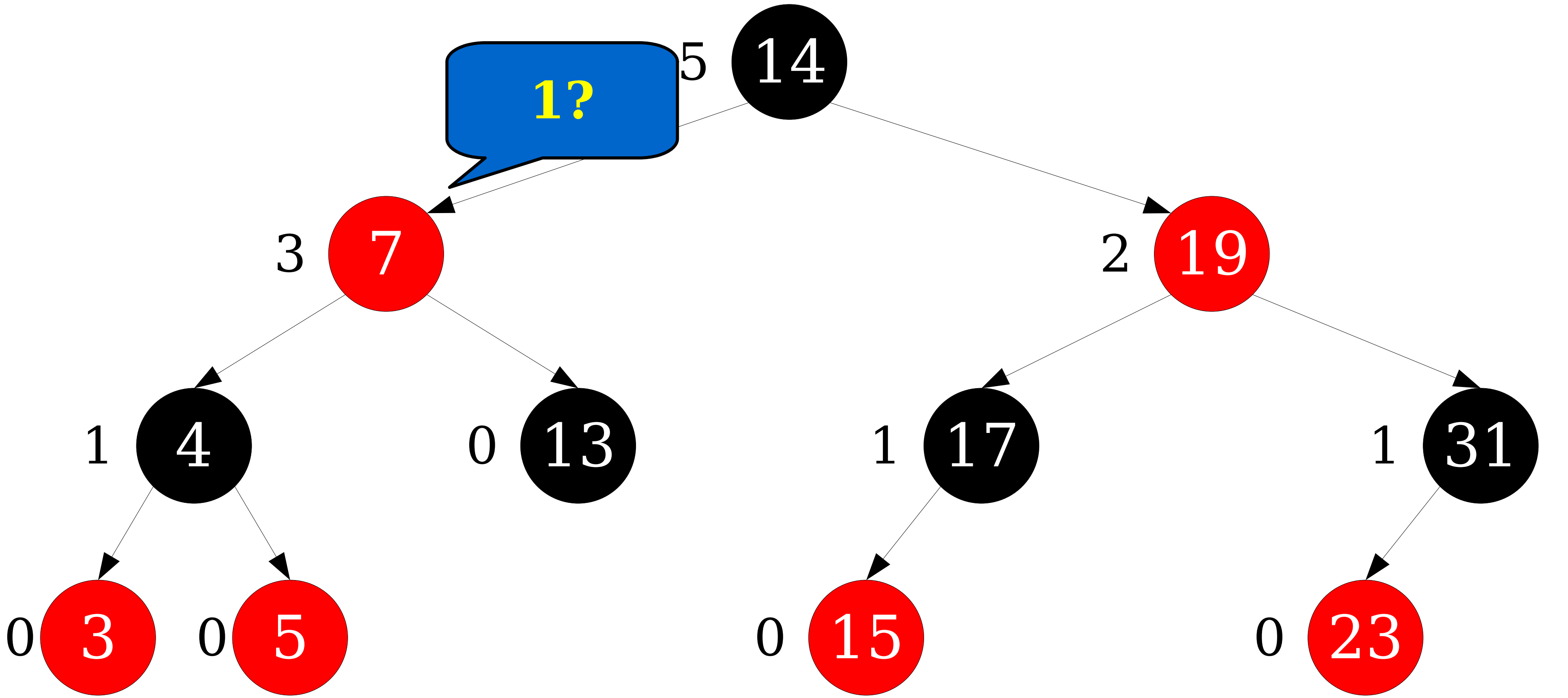

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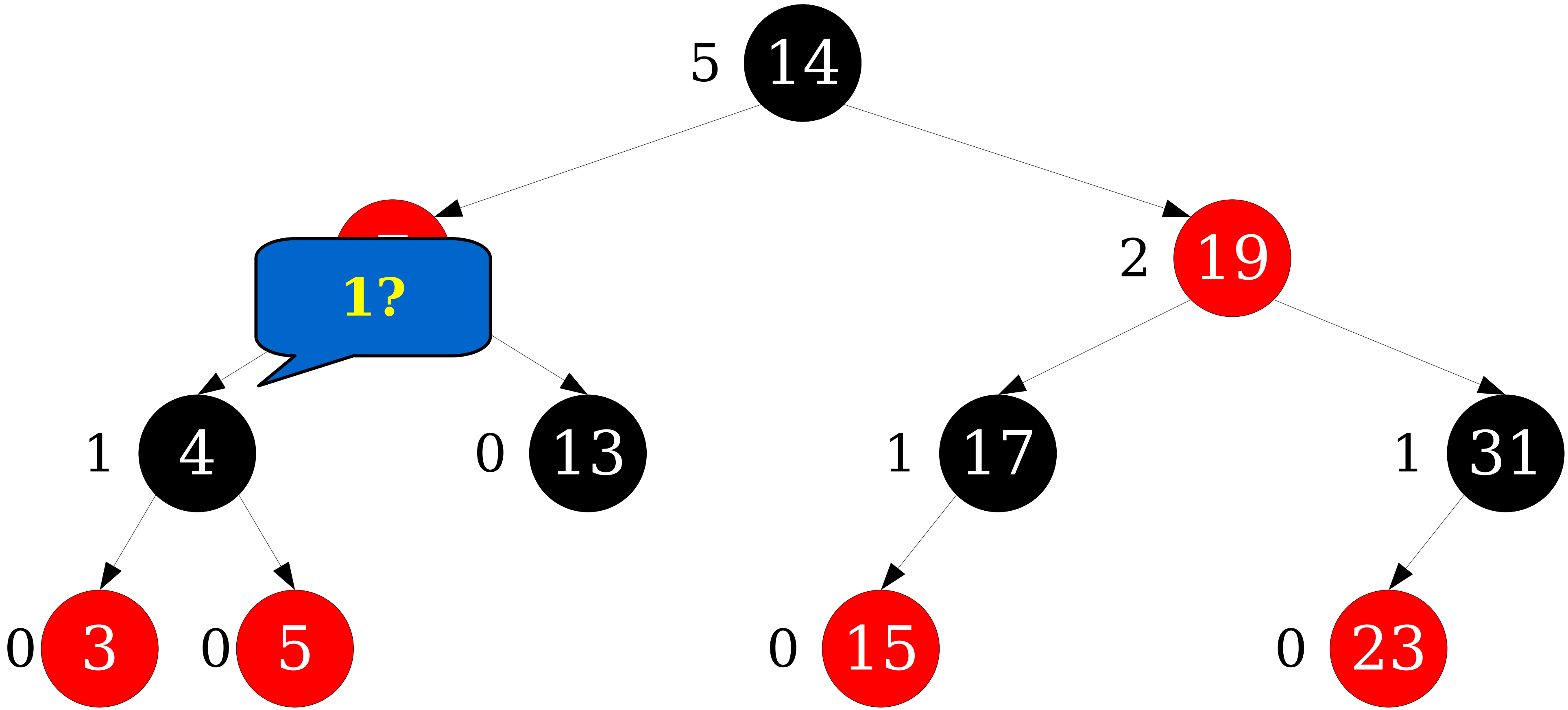

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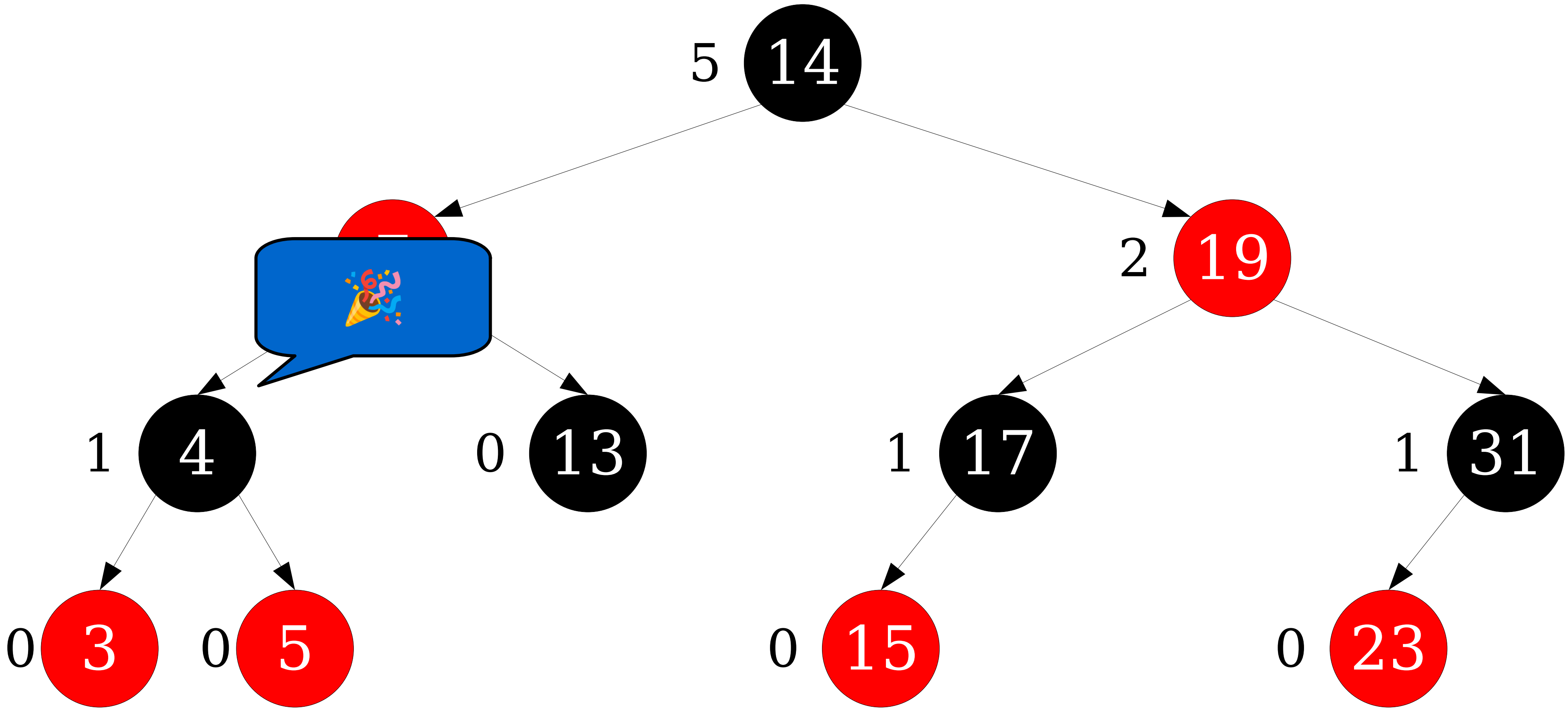

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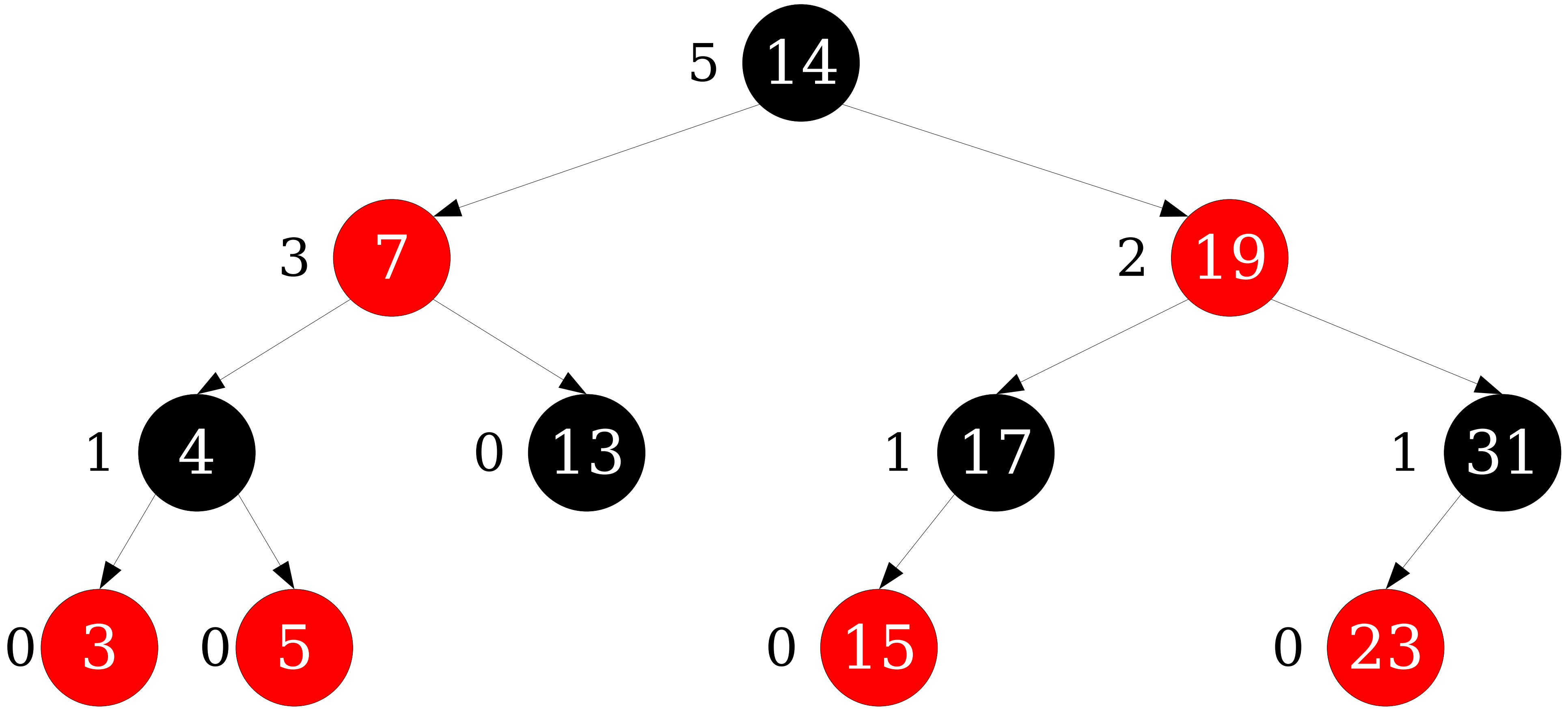

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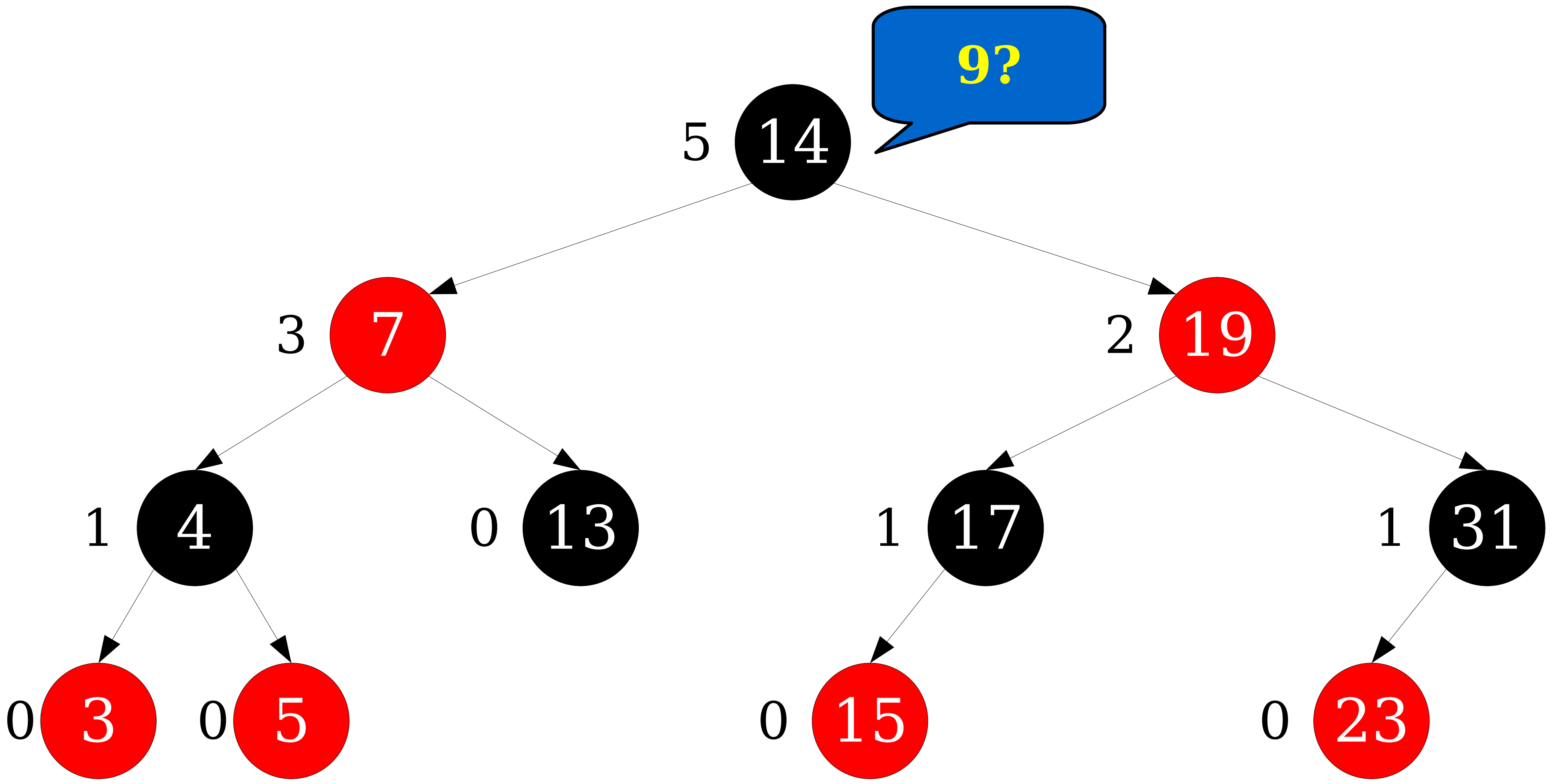

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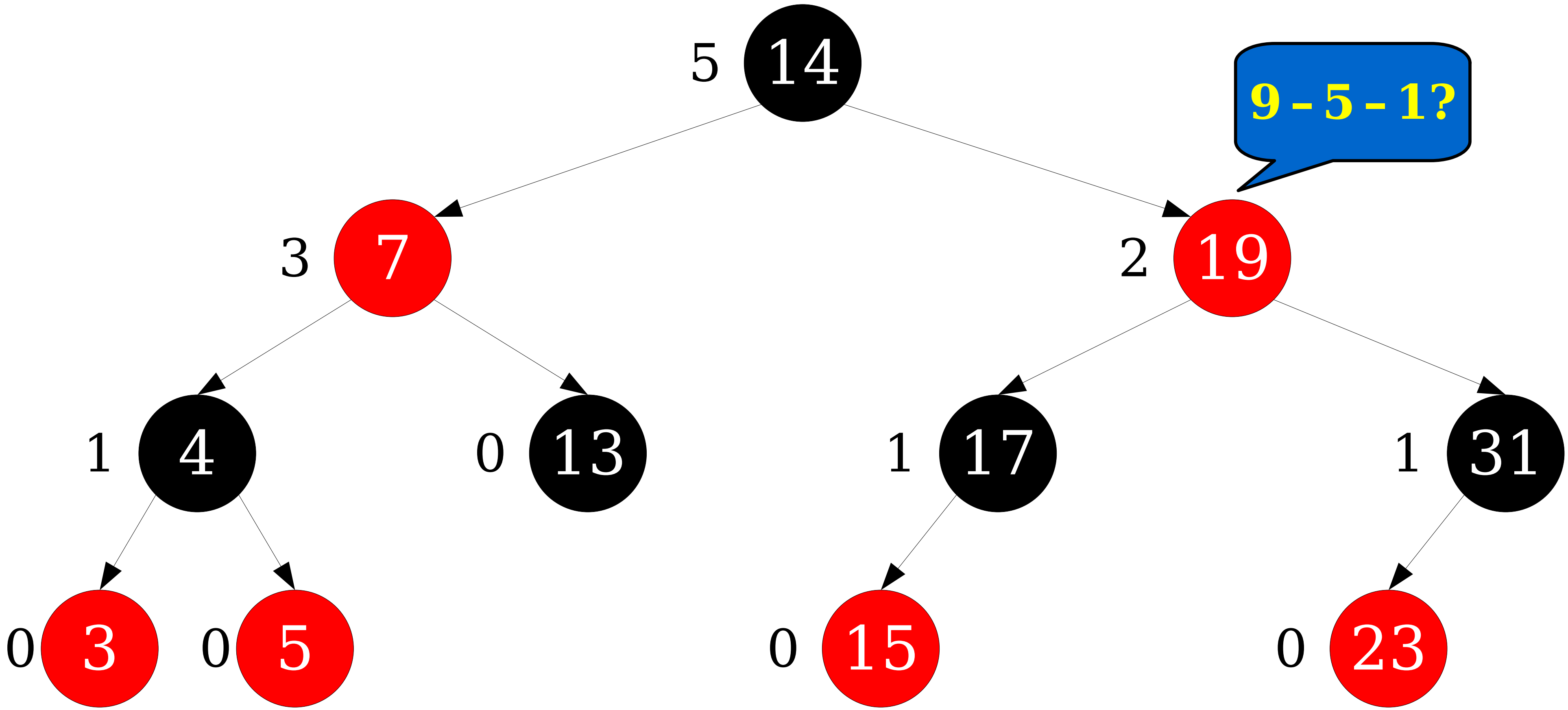

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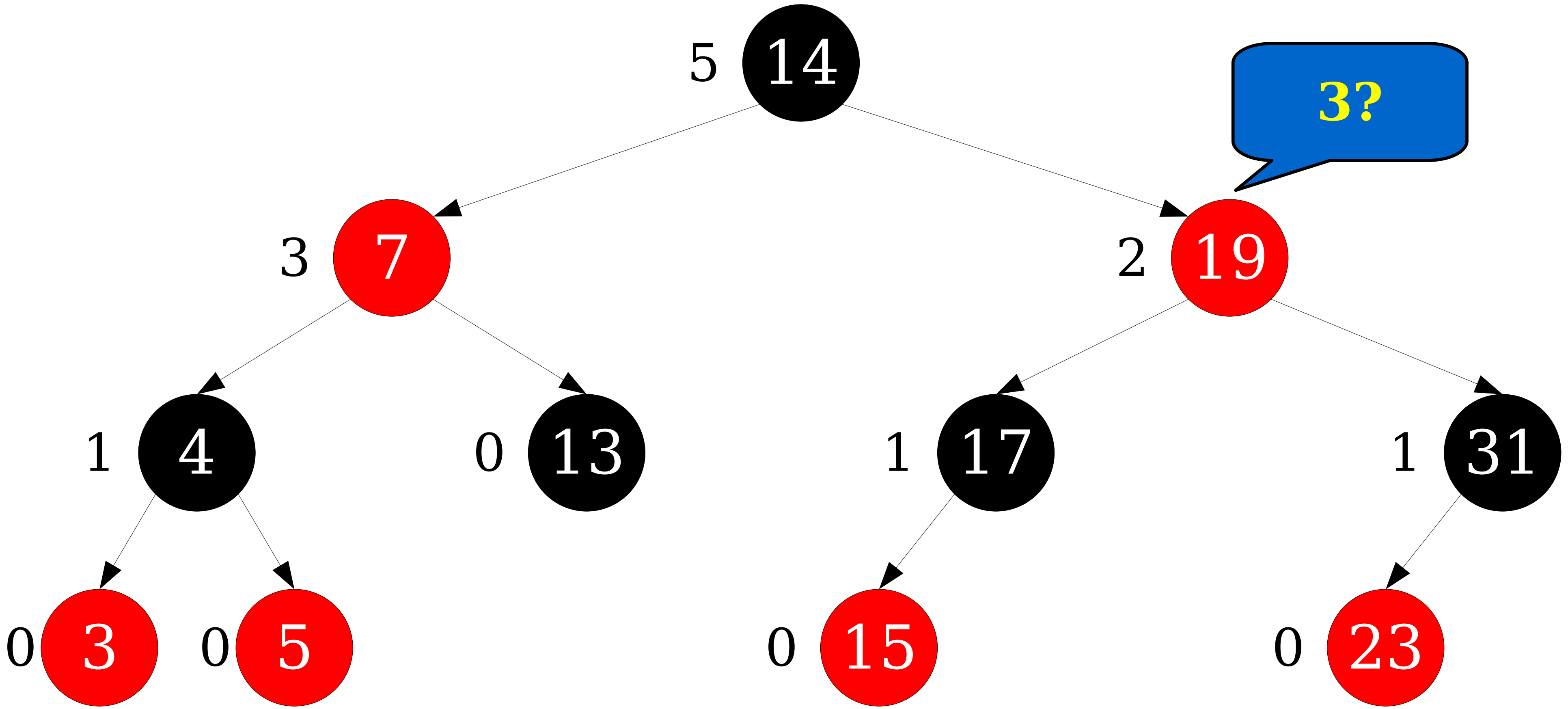

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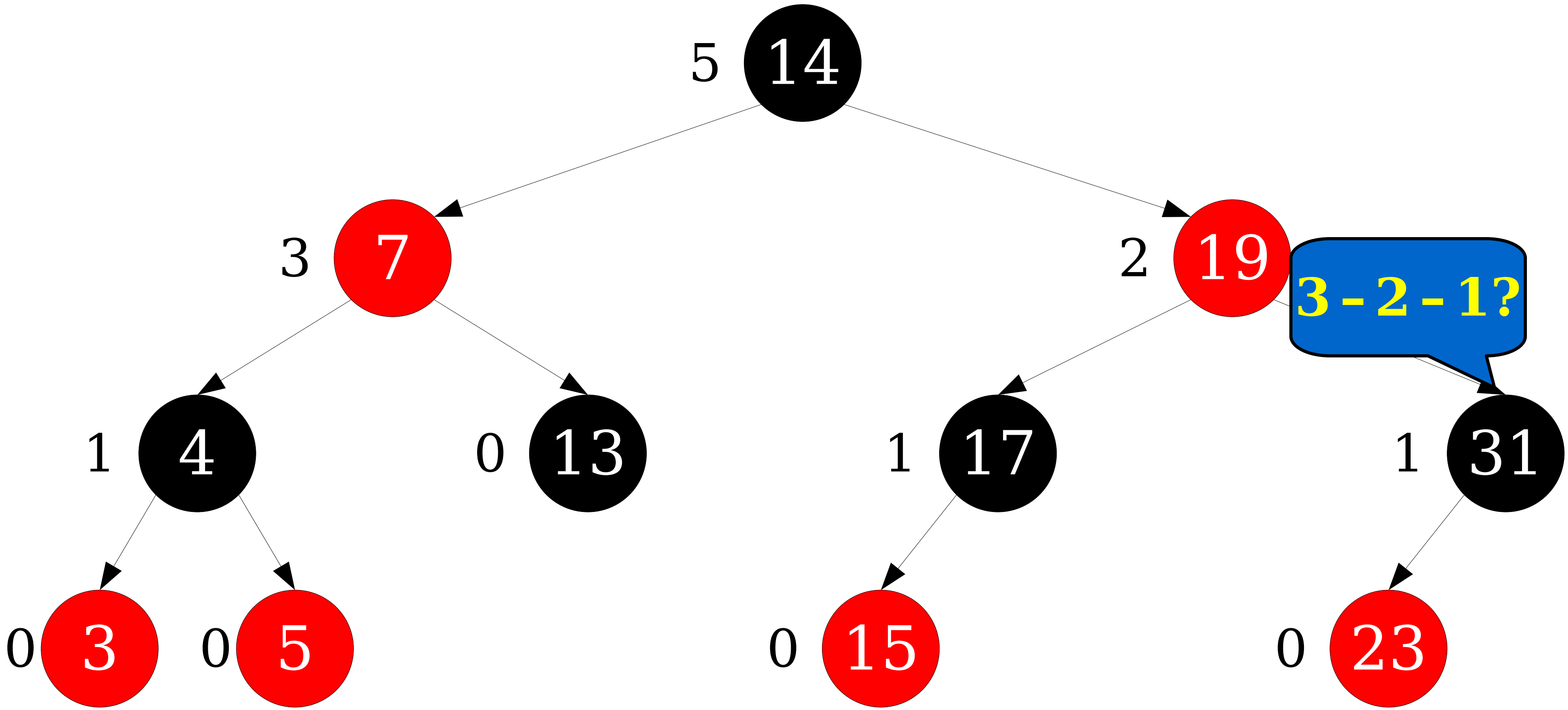

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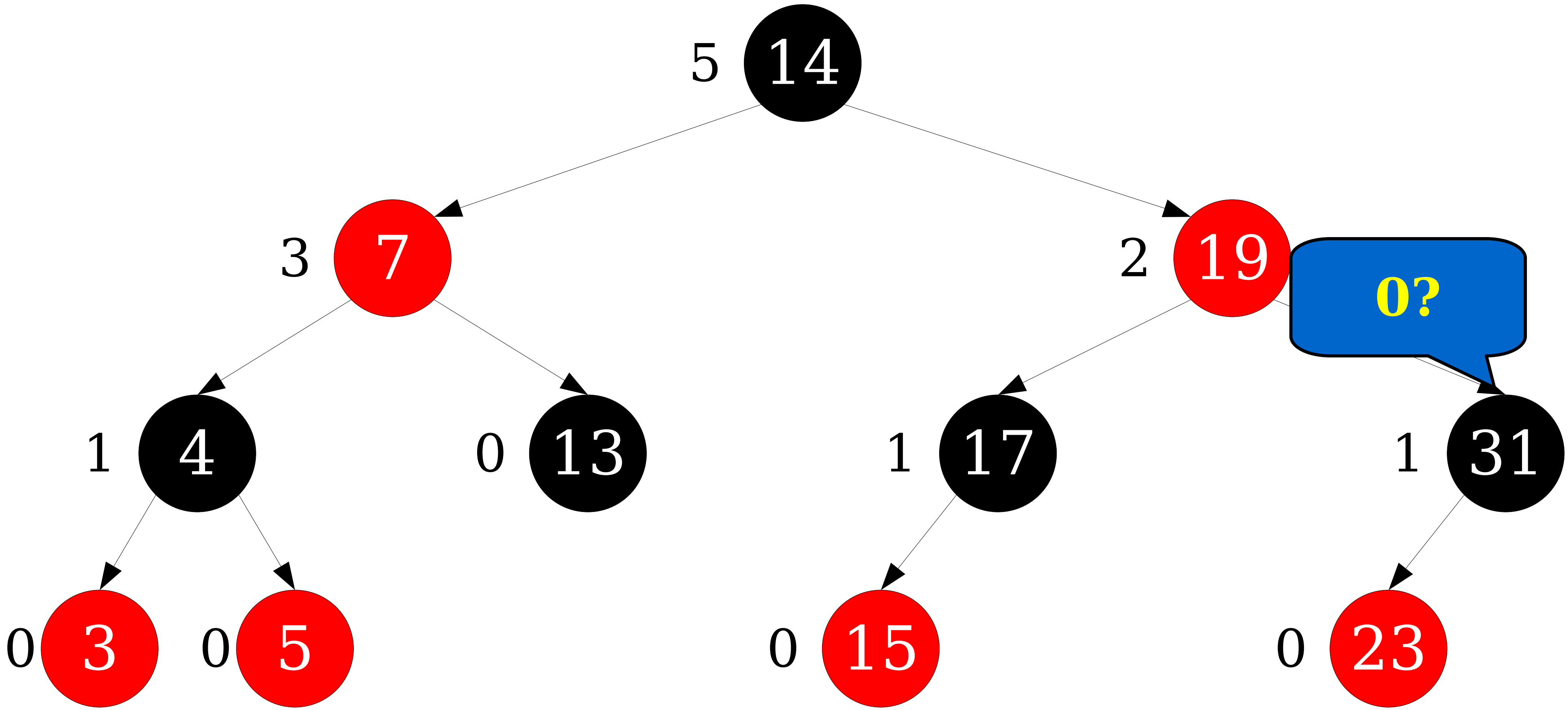

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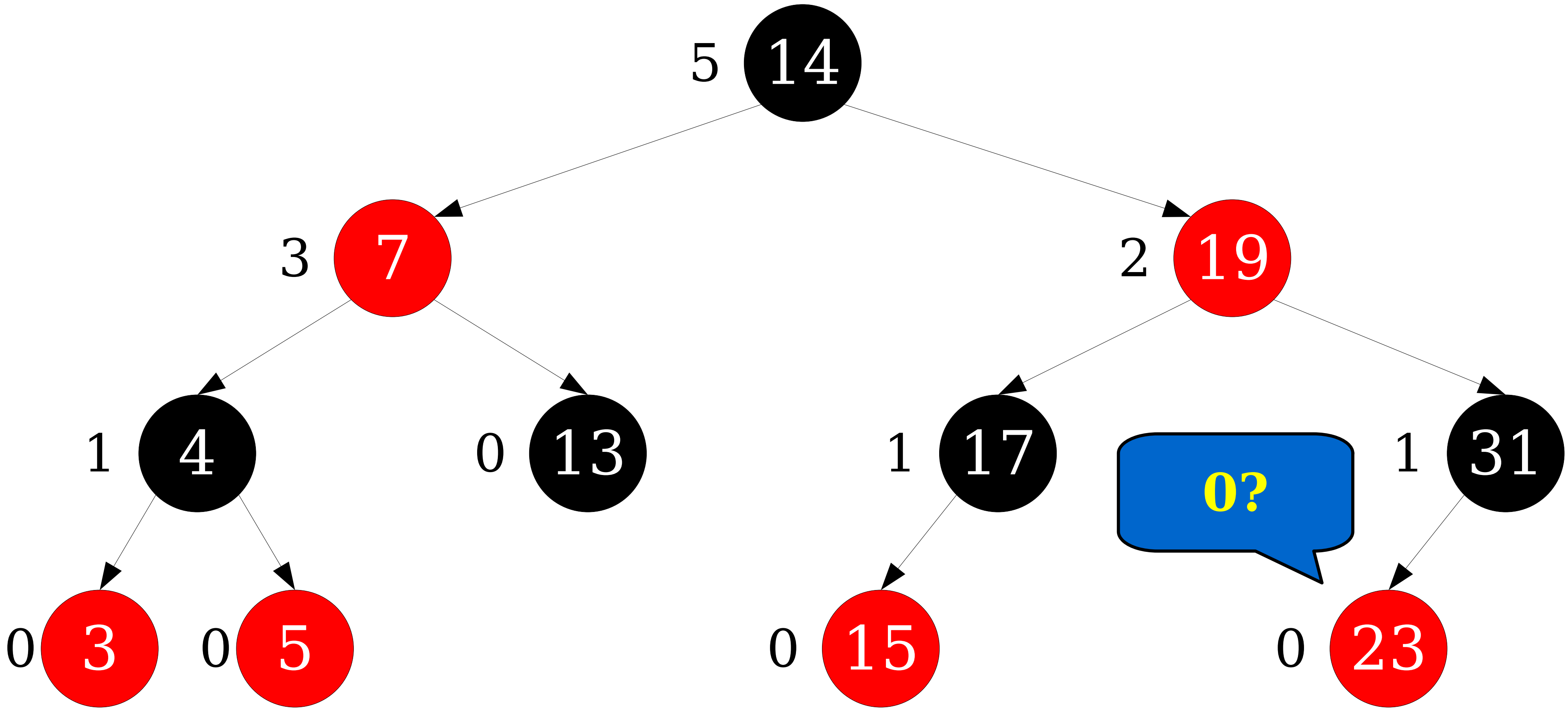
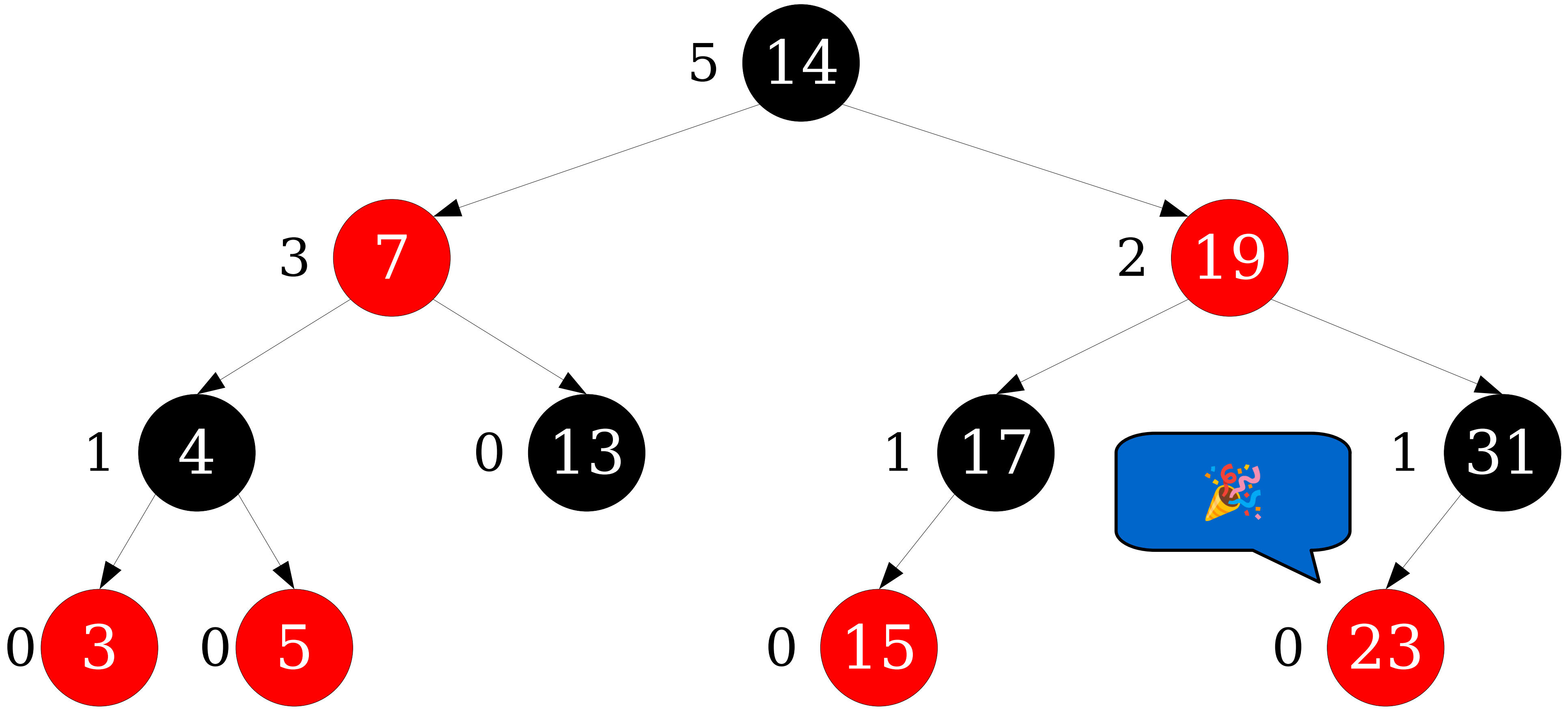


Dynamic Selection



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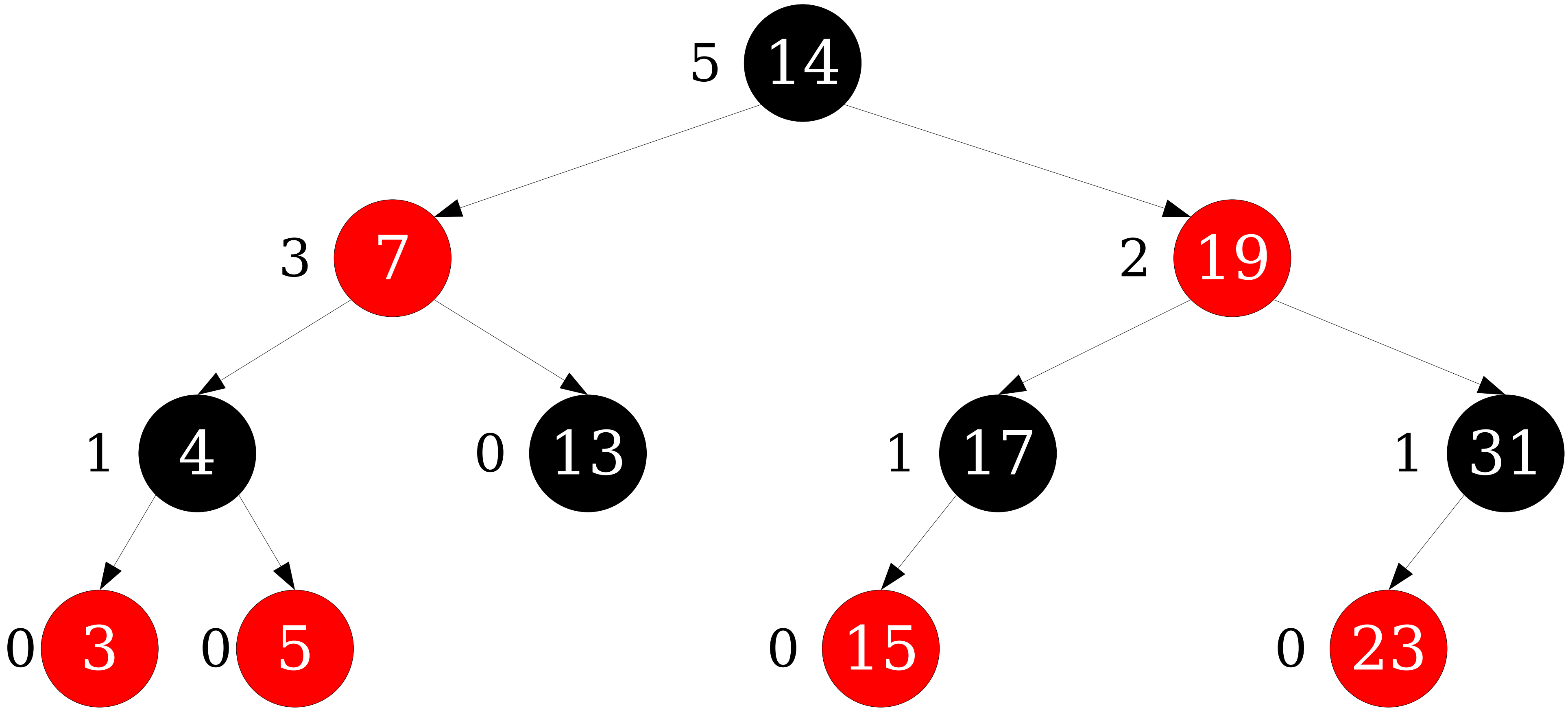

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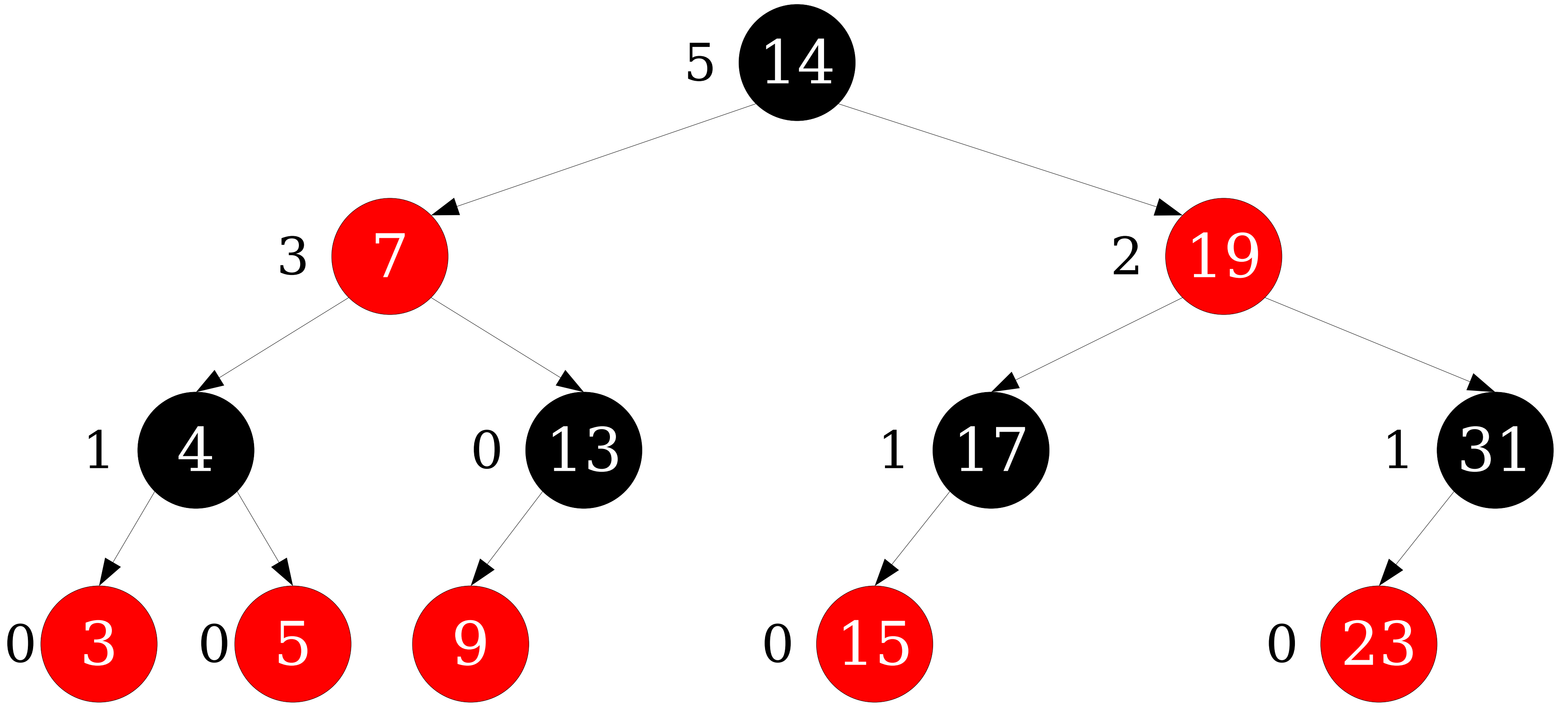

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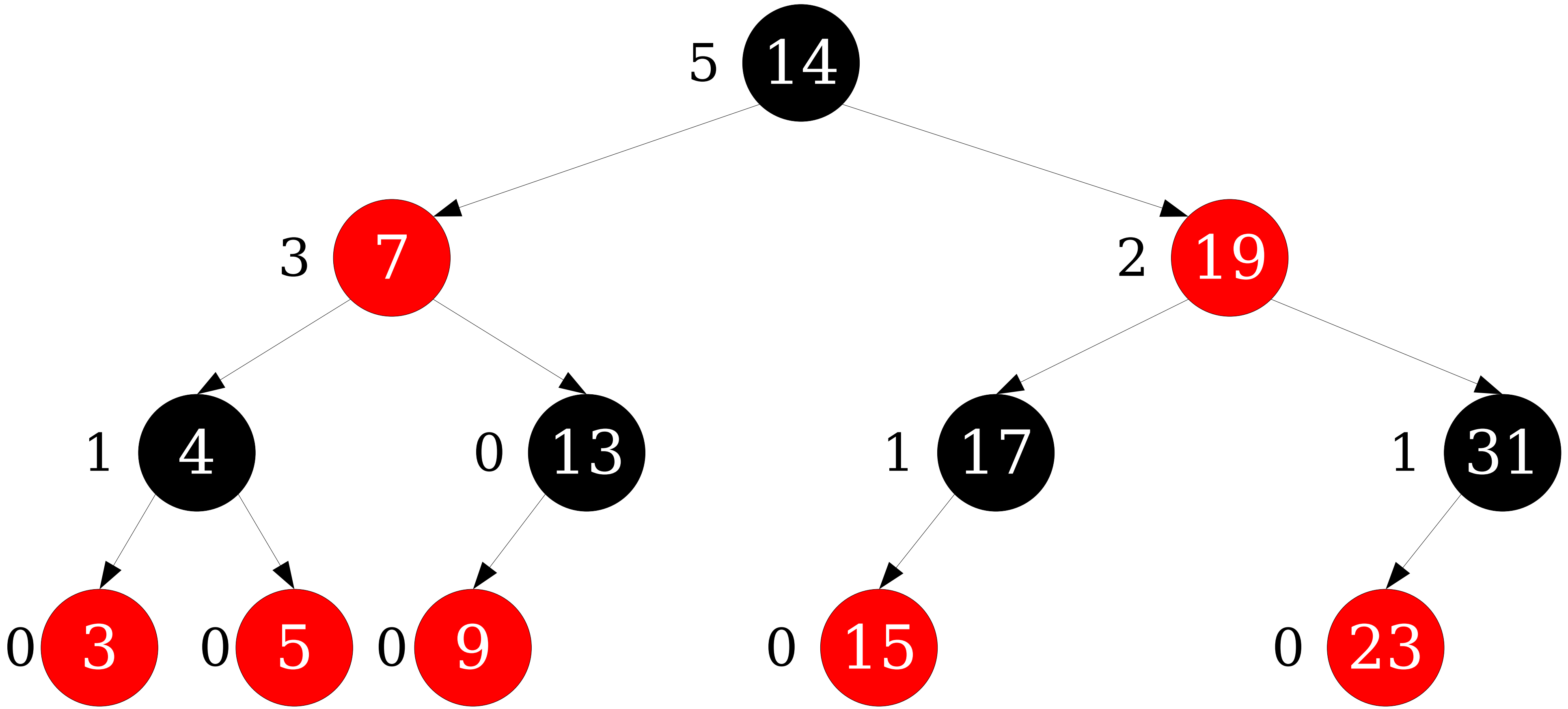
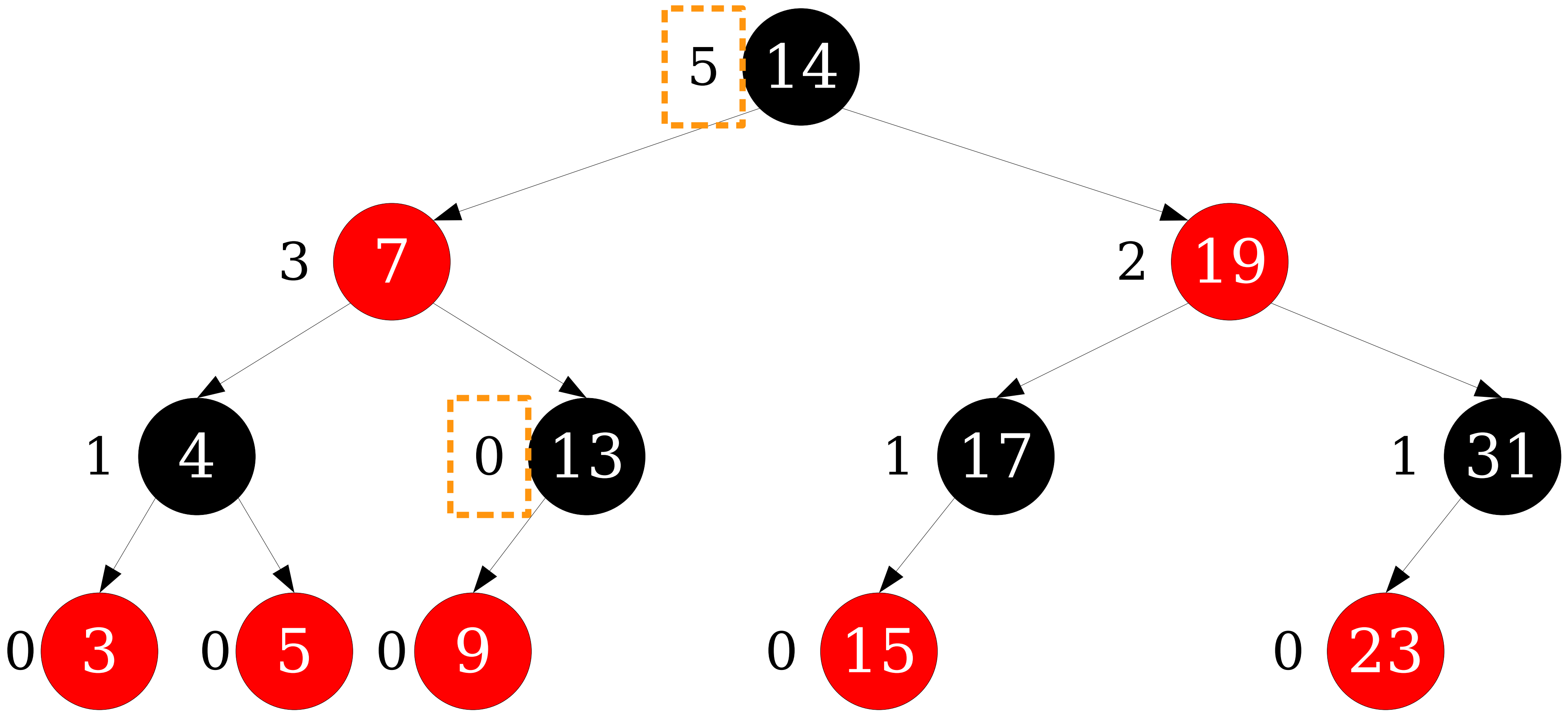
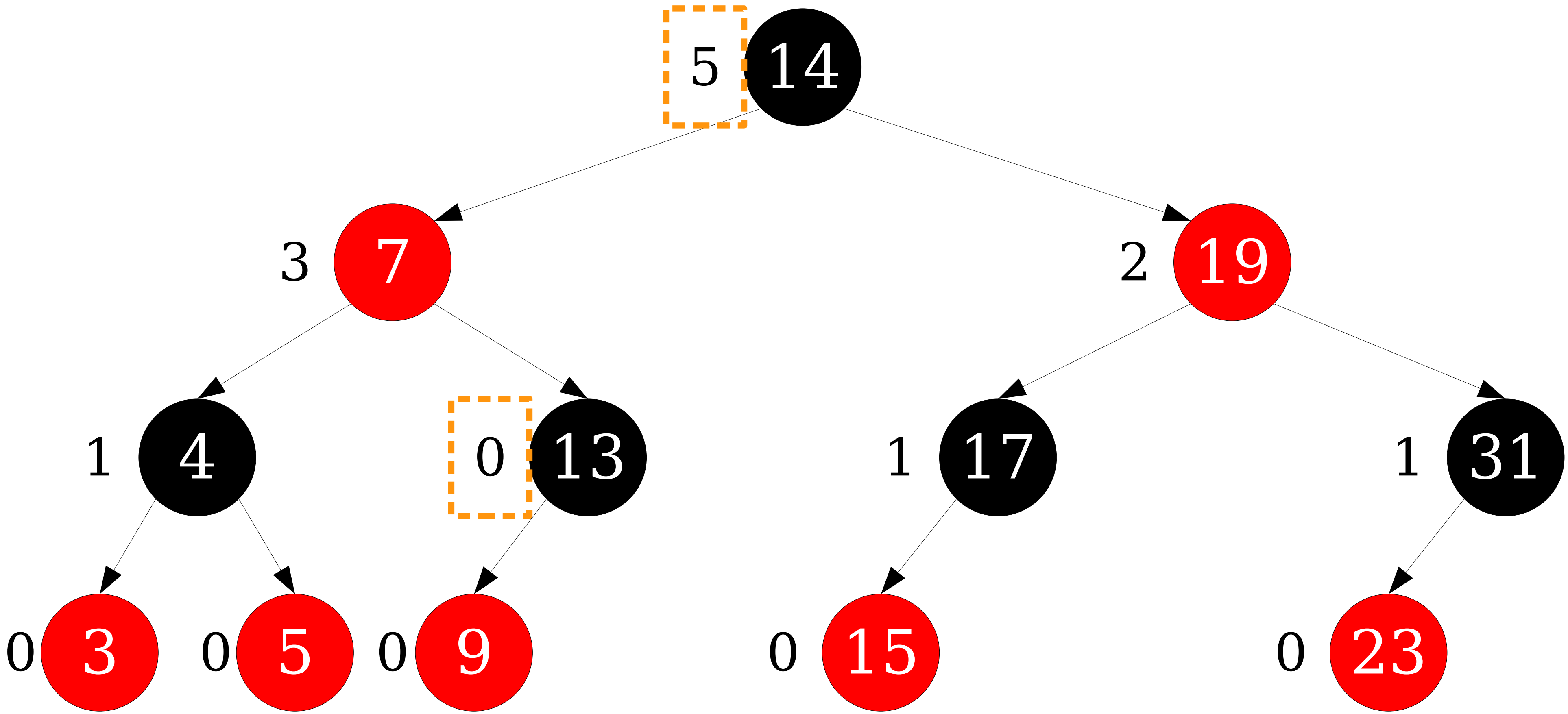


Dynamic Selection

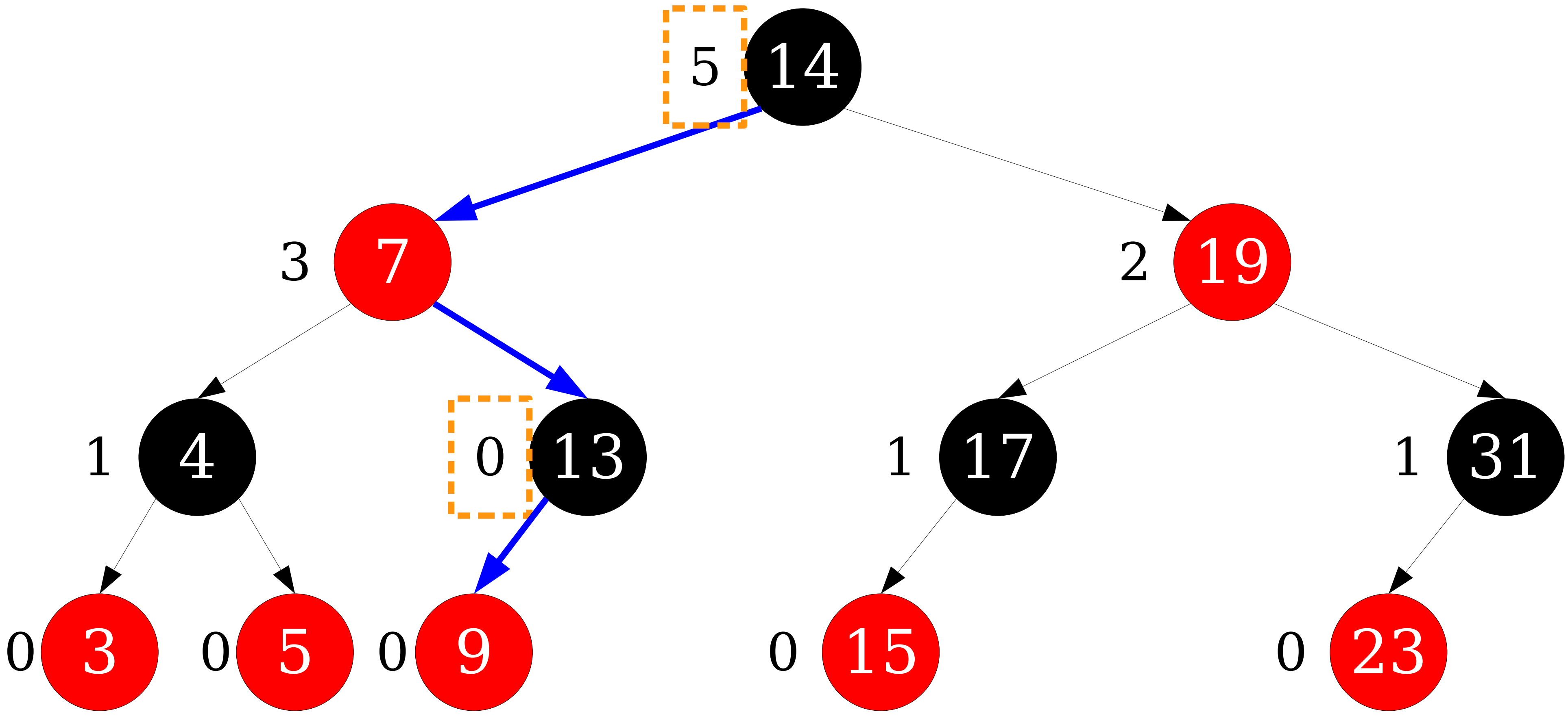


Dynamic Selection



We only update values on nodes that gained a new key in their left subtree. And there are only O(log *n*) of these!

Dynamic Selection



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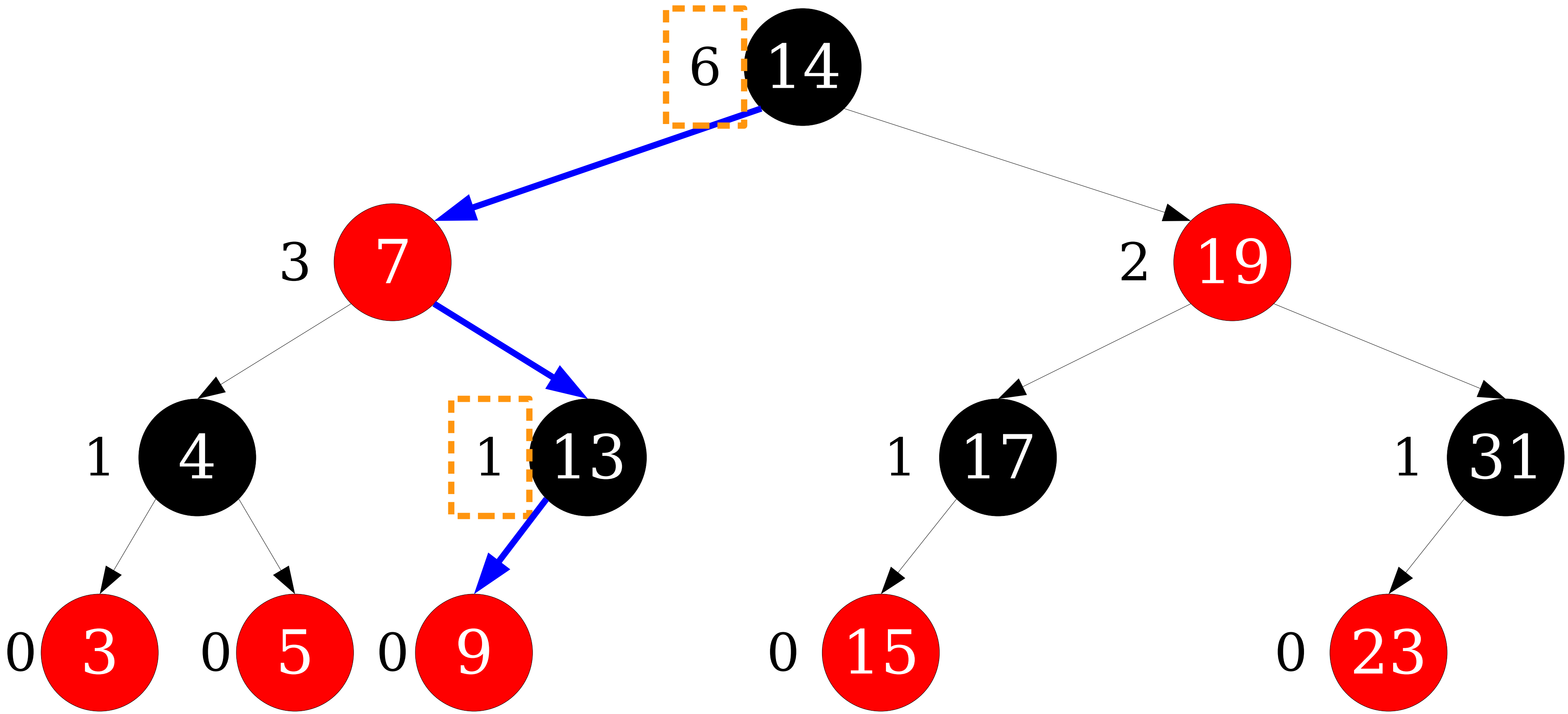
Dynamic Selection



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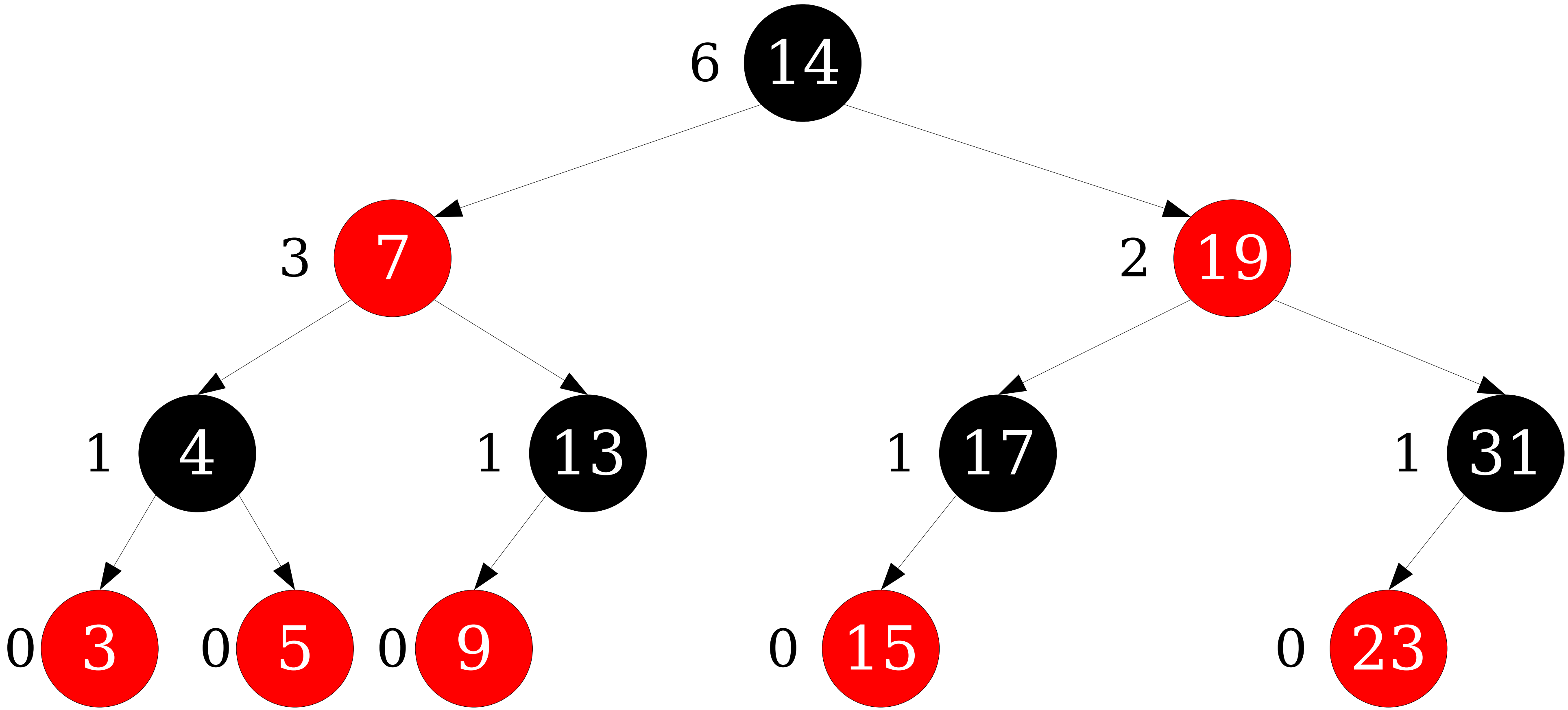
Dynamic Selection



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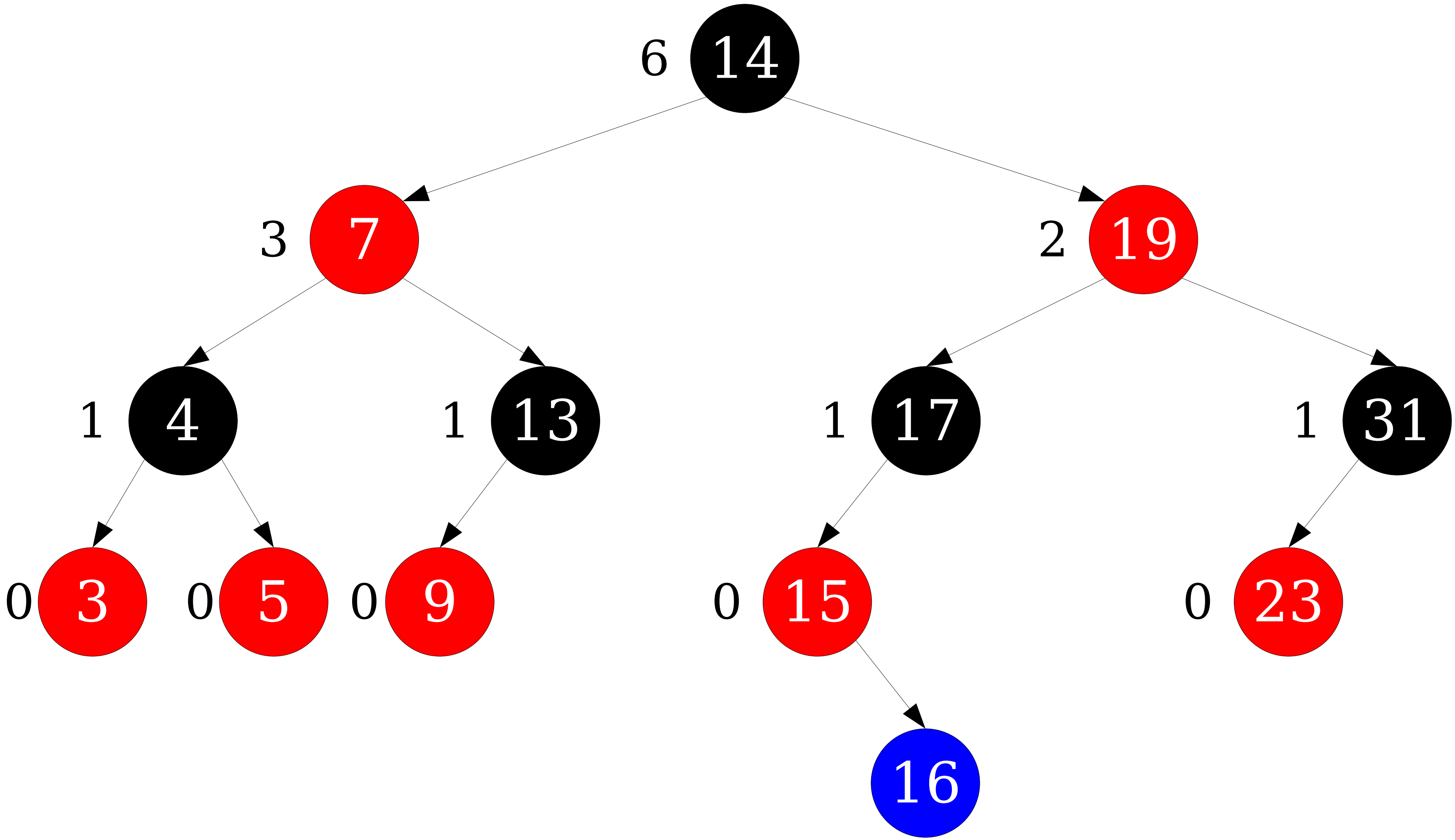

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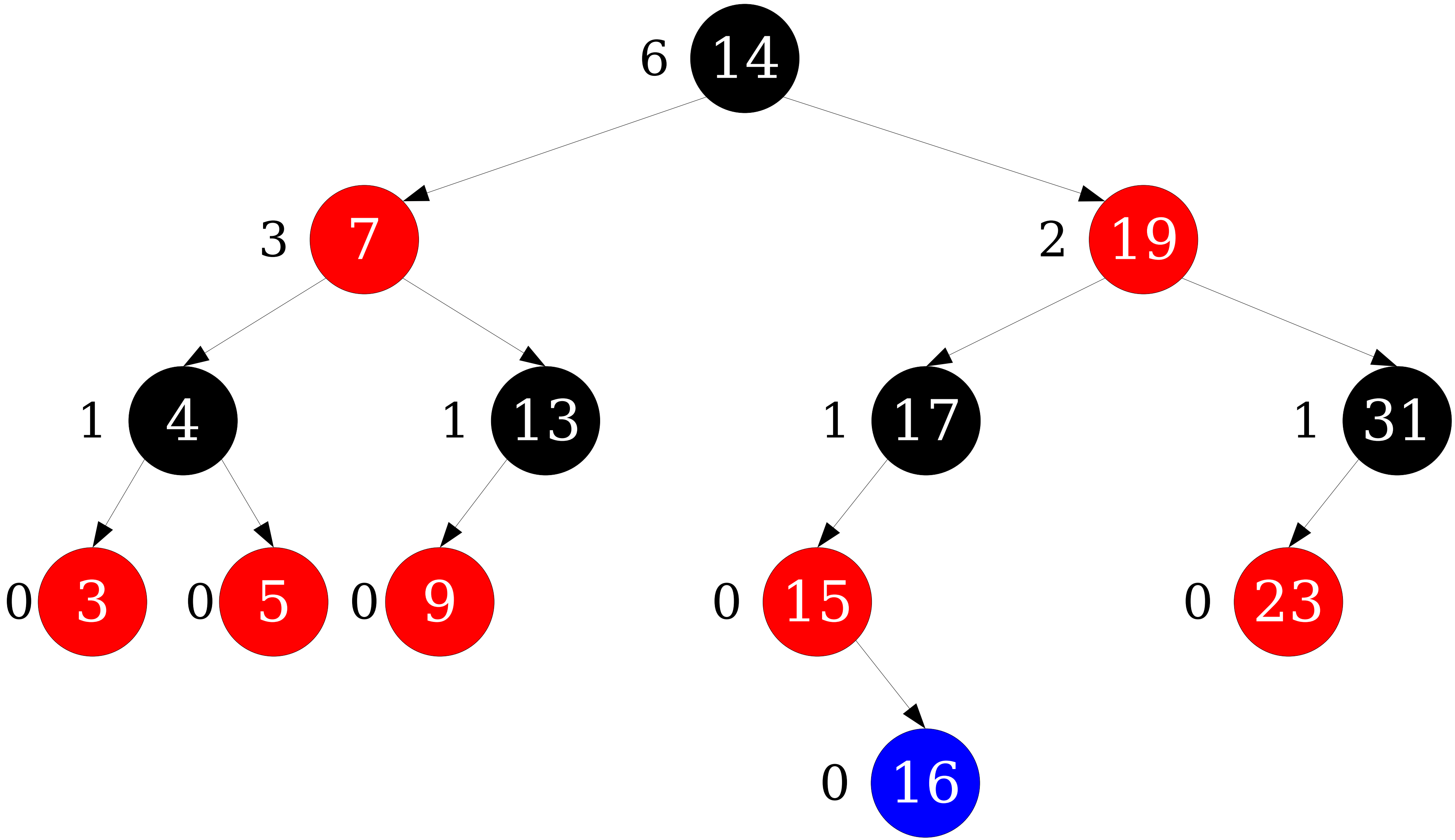

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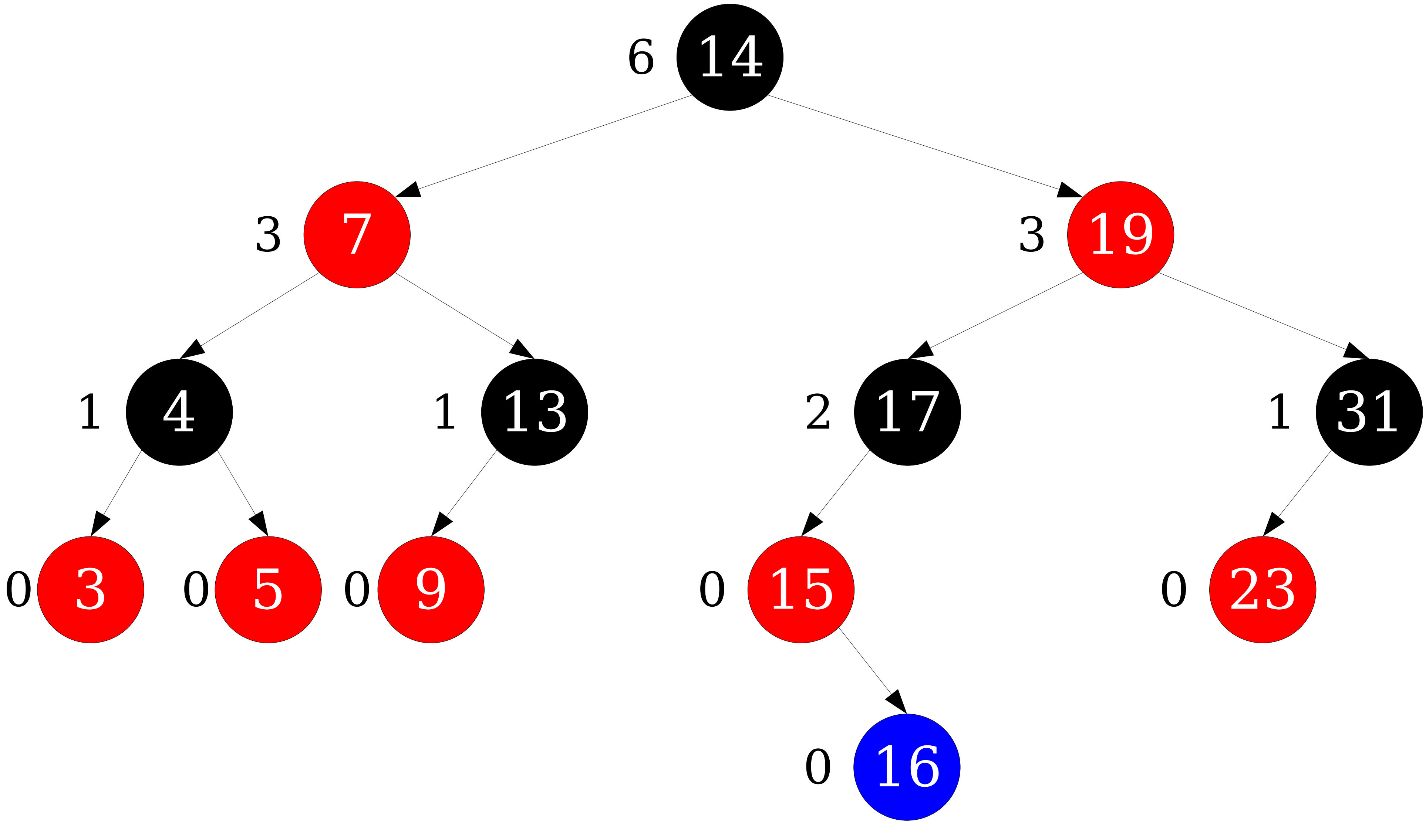

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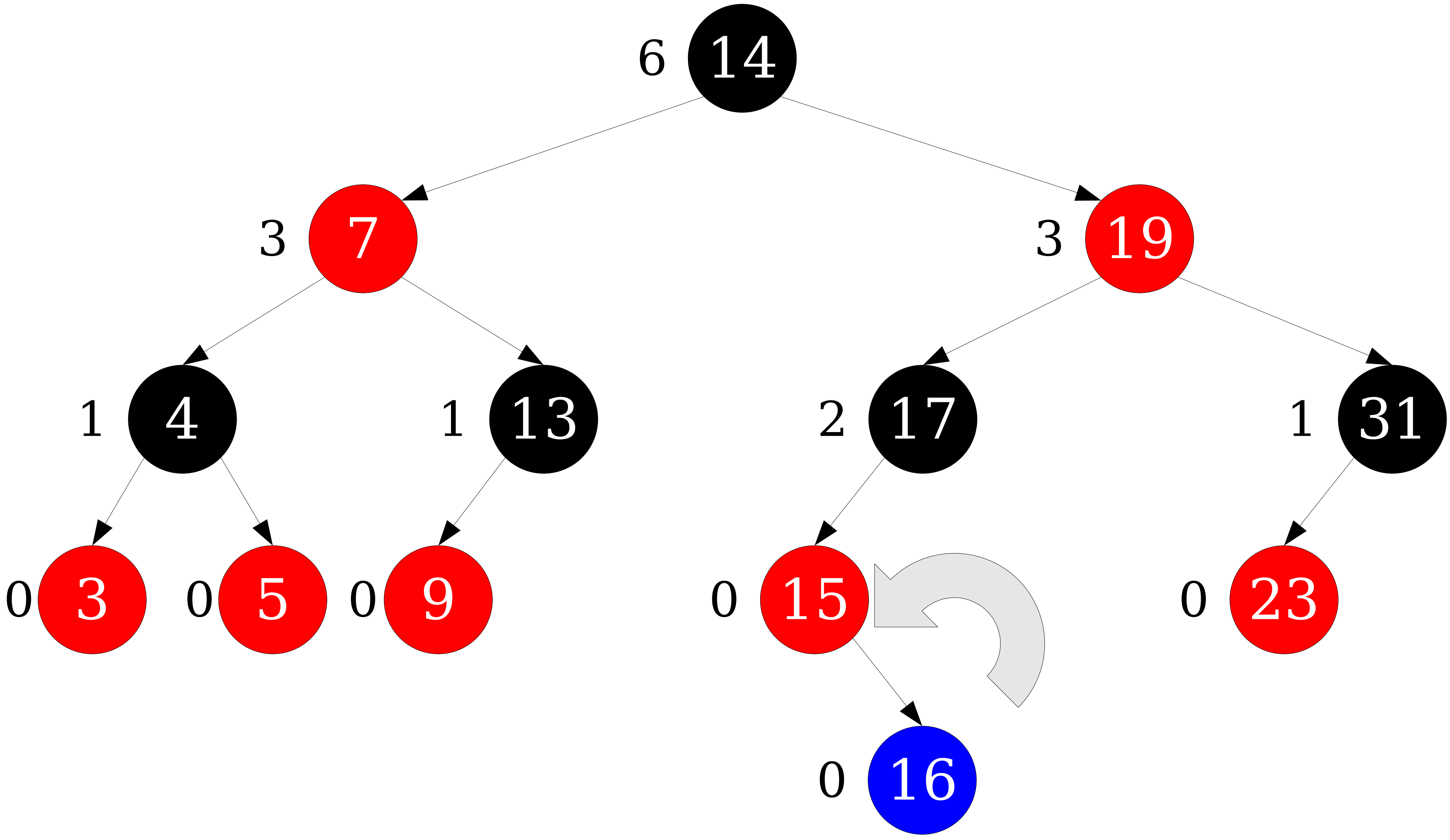

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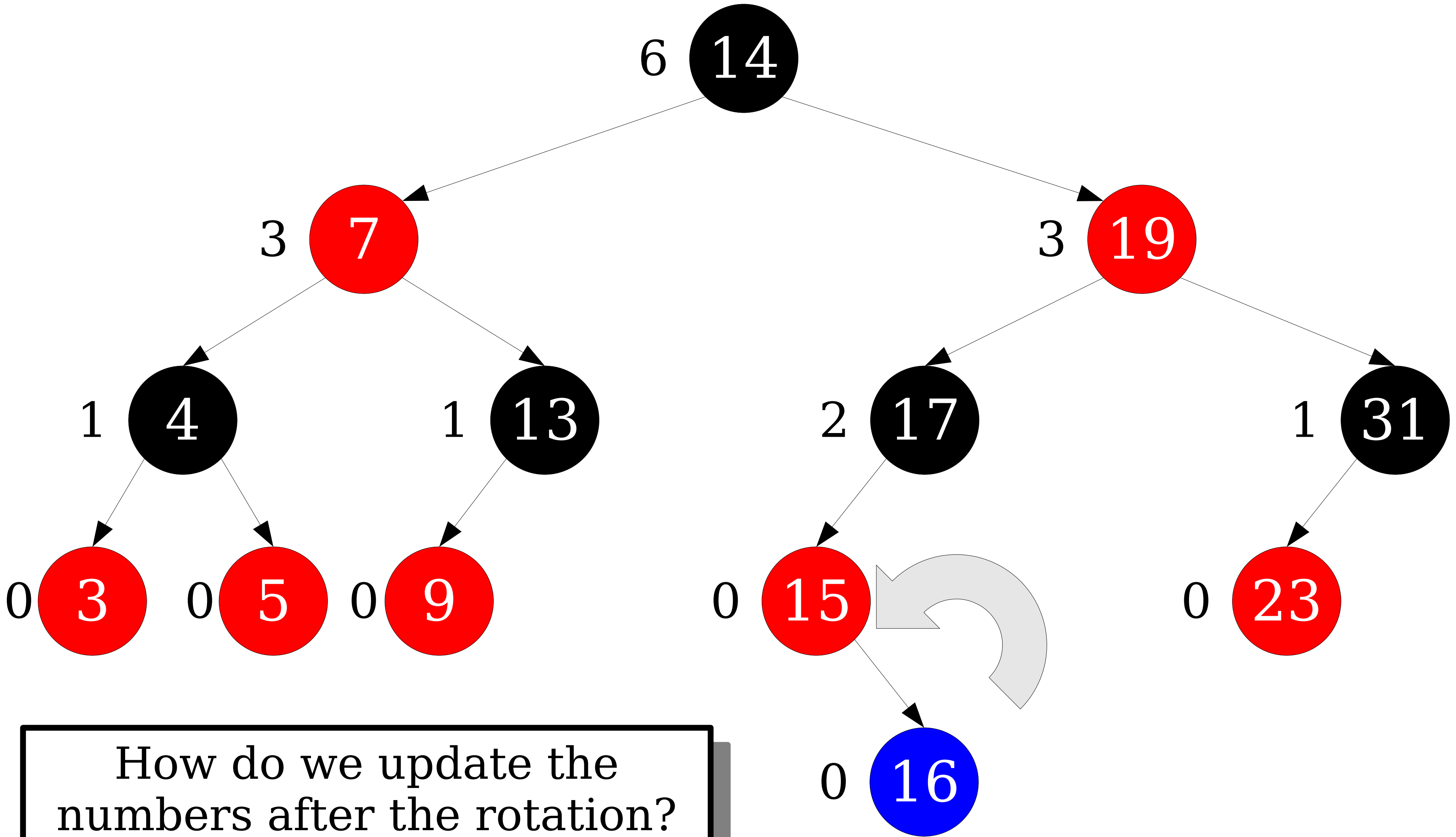
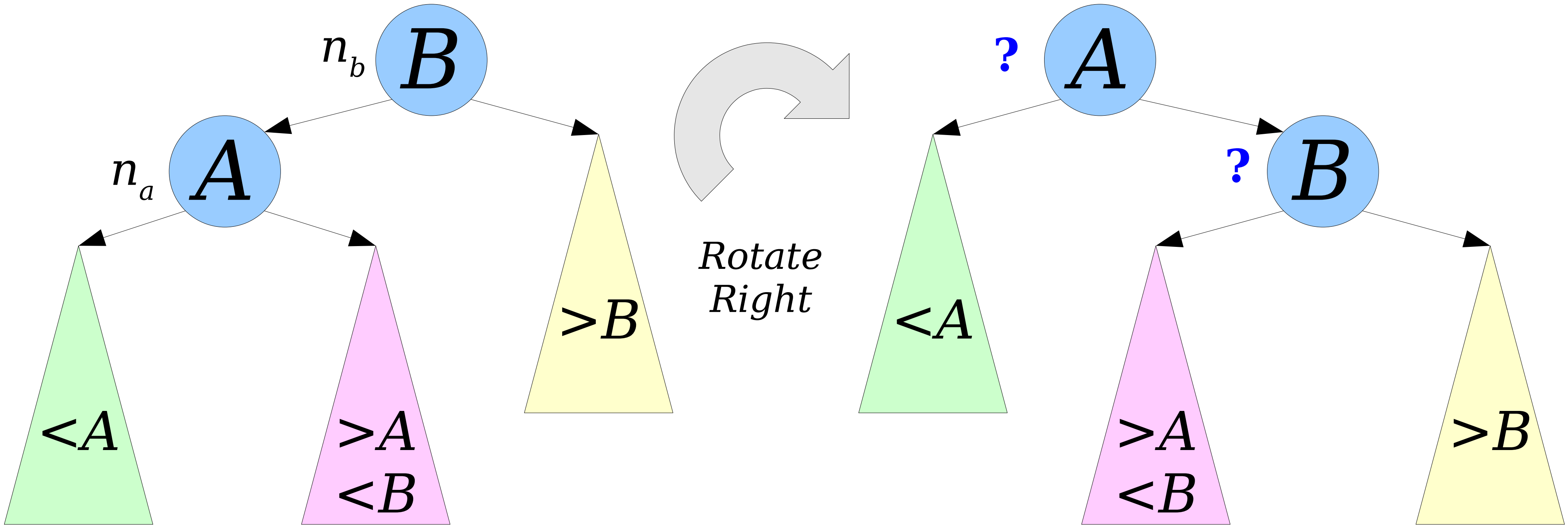
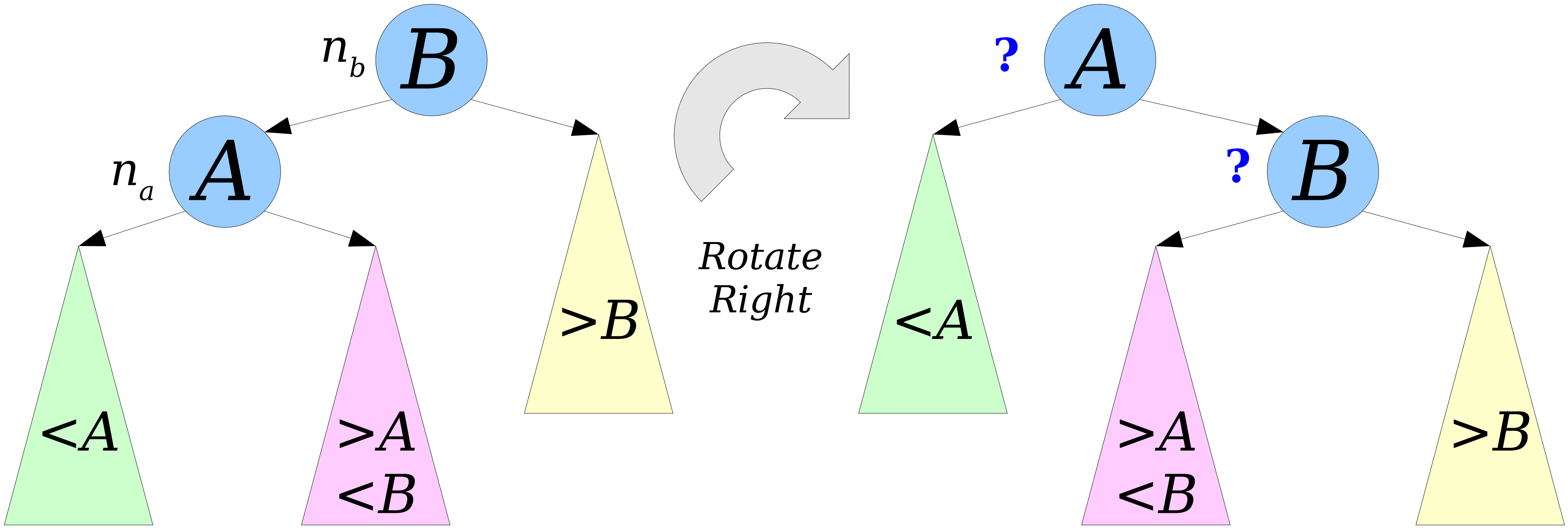


How do we update the numbers after the rotation?

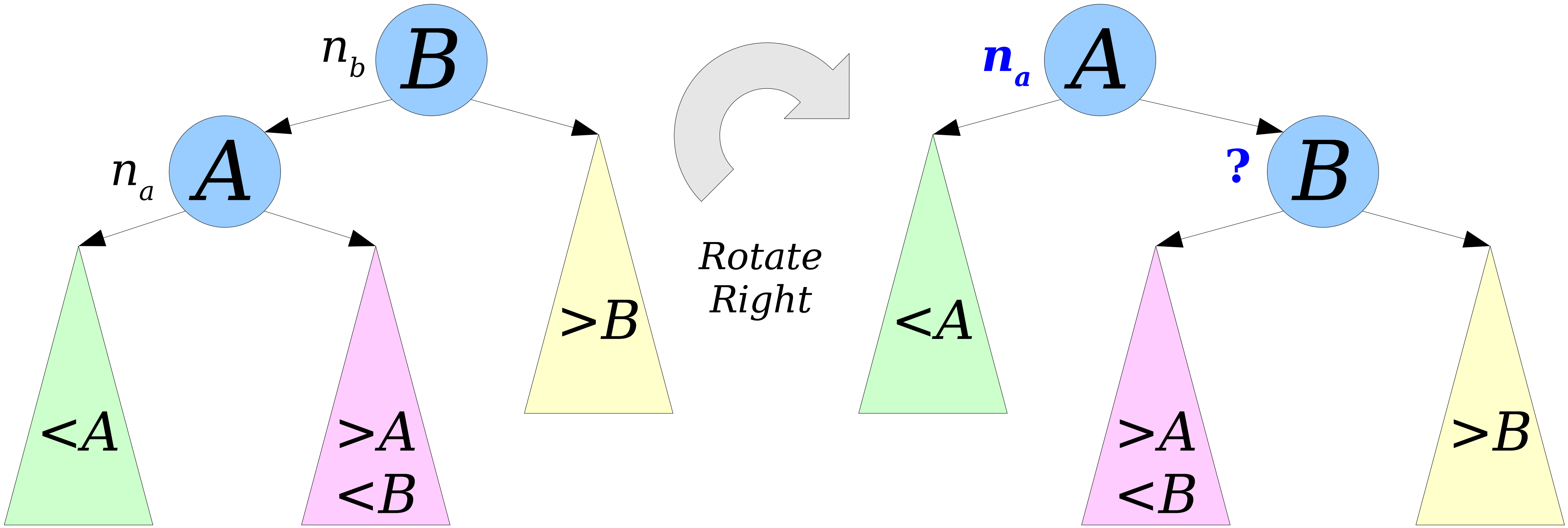


What numbers go here? Answer at

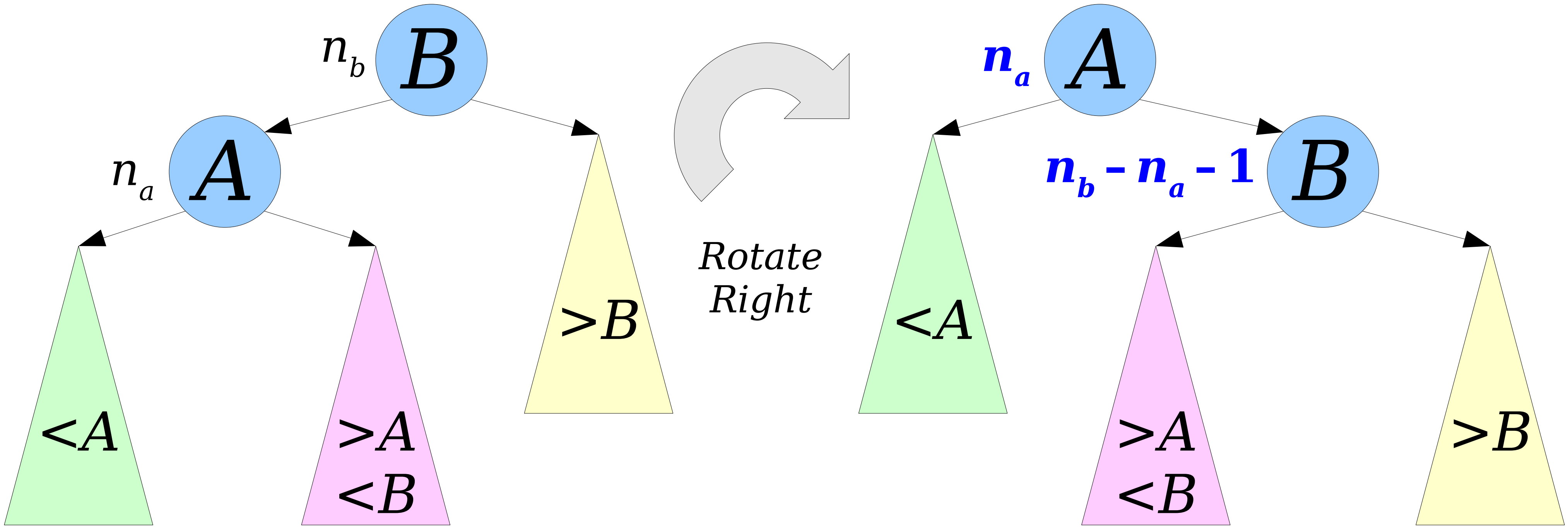
https://cs166.stanford.edu/pollev

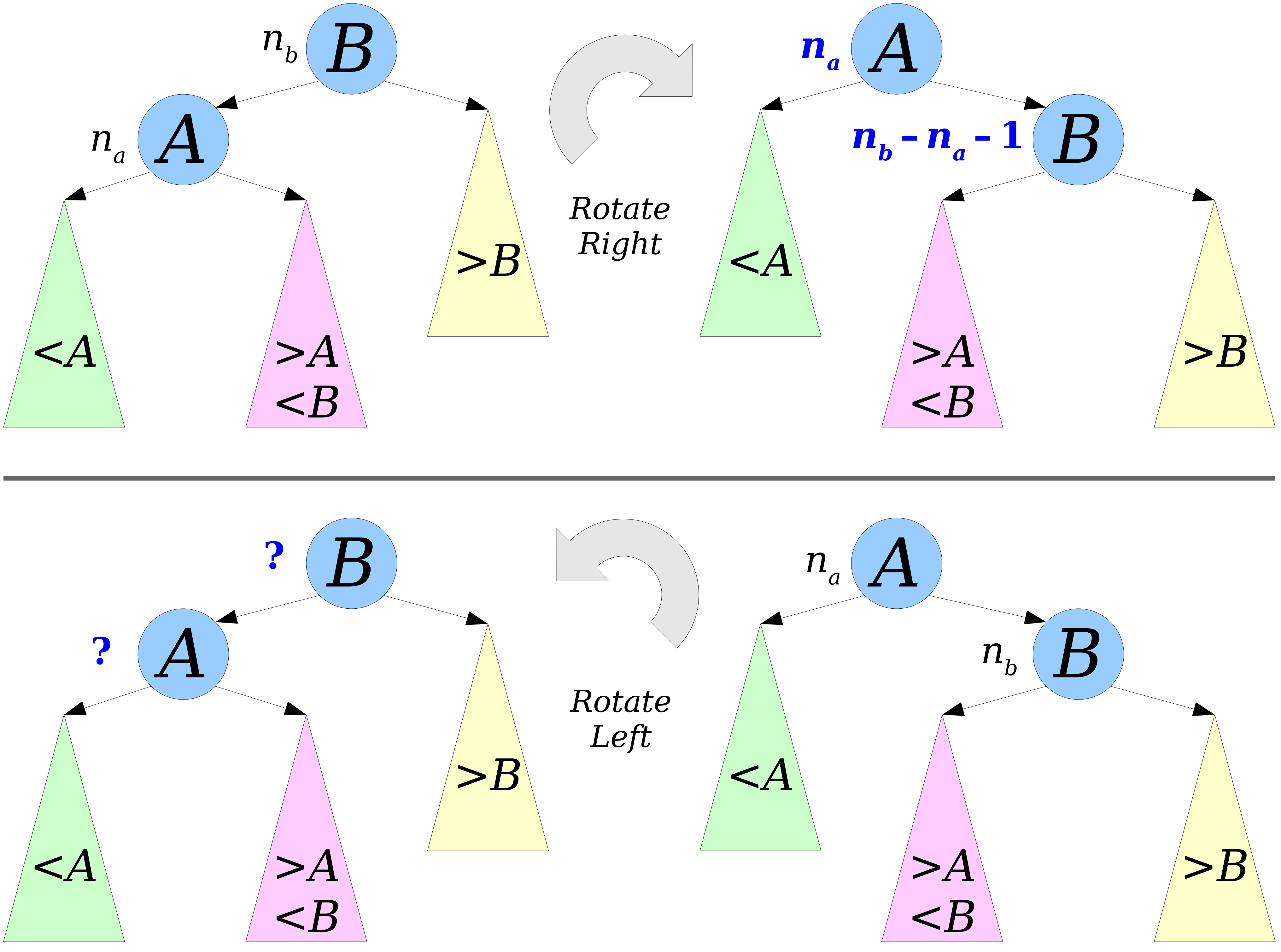


n_b
B
n_a
A
<A
>A
<B
>B
Rotate
Right
?
A
?
B
<A
>A
<B
>B

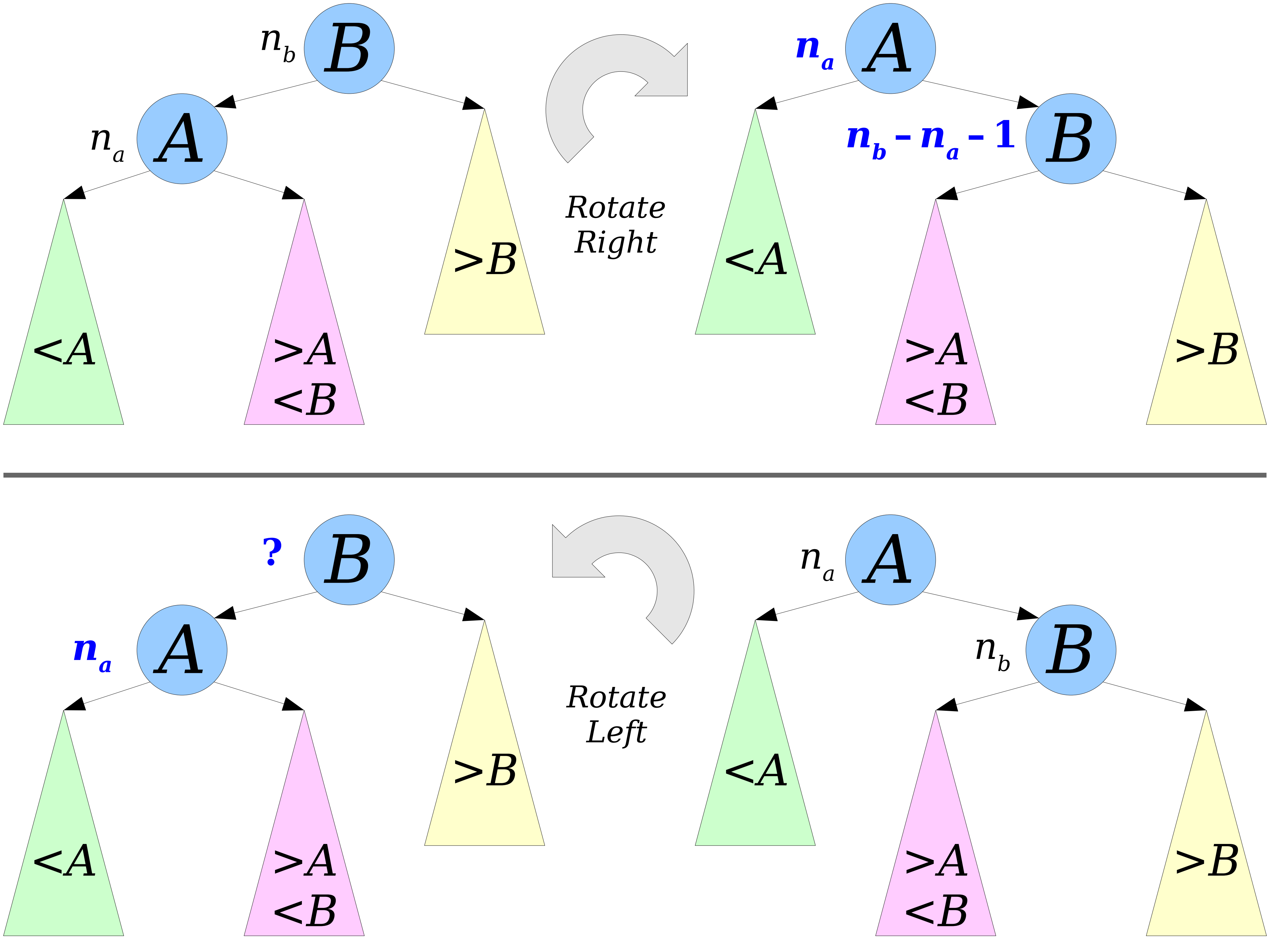


n_b
B
n_a
A
<A
>A
<B
>B
Rotate
Right
n_a
A
?
B
<A
>A
<B
>B

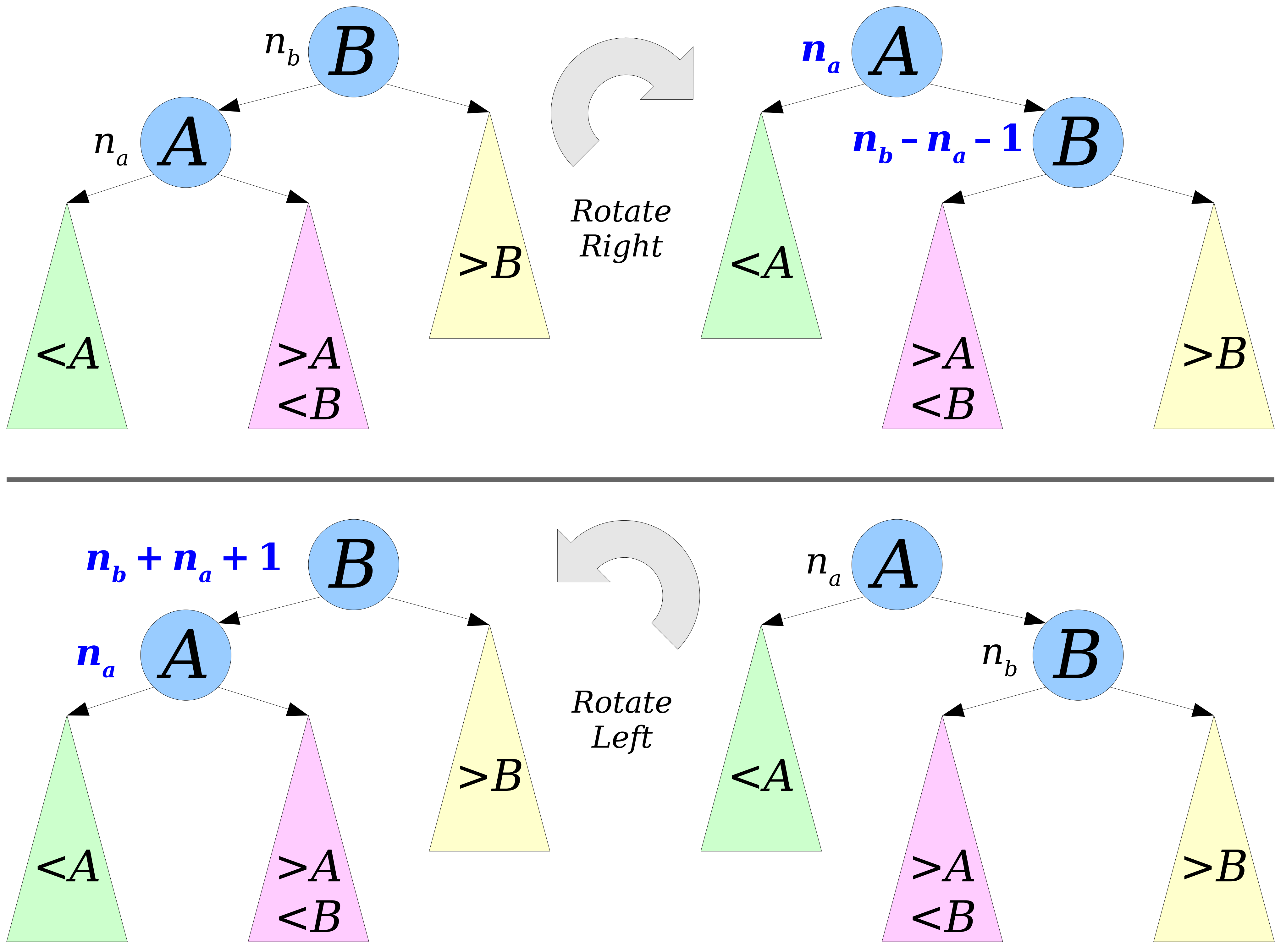

n_b
B
n_a
A
<A
>A
<B
>B
Rotate
Right
n_a
A
$n_b - n_a - 1$
B
<A
>A
<B
>B



n_b
B
n_a
A
<A
>A
<B
>B
Rotate
Right
n_a
A
$n_b - n_a - 1$
B
<A
>A
<B
>B
?
B
?
A
<A
>A
<B
>B
Rotate
Left
n_a
A
n_b
B
<A
>A
<B
>B

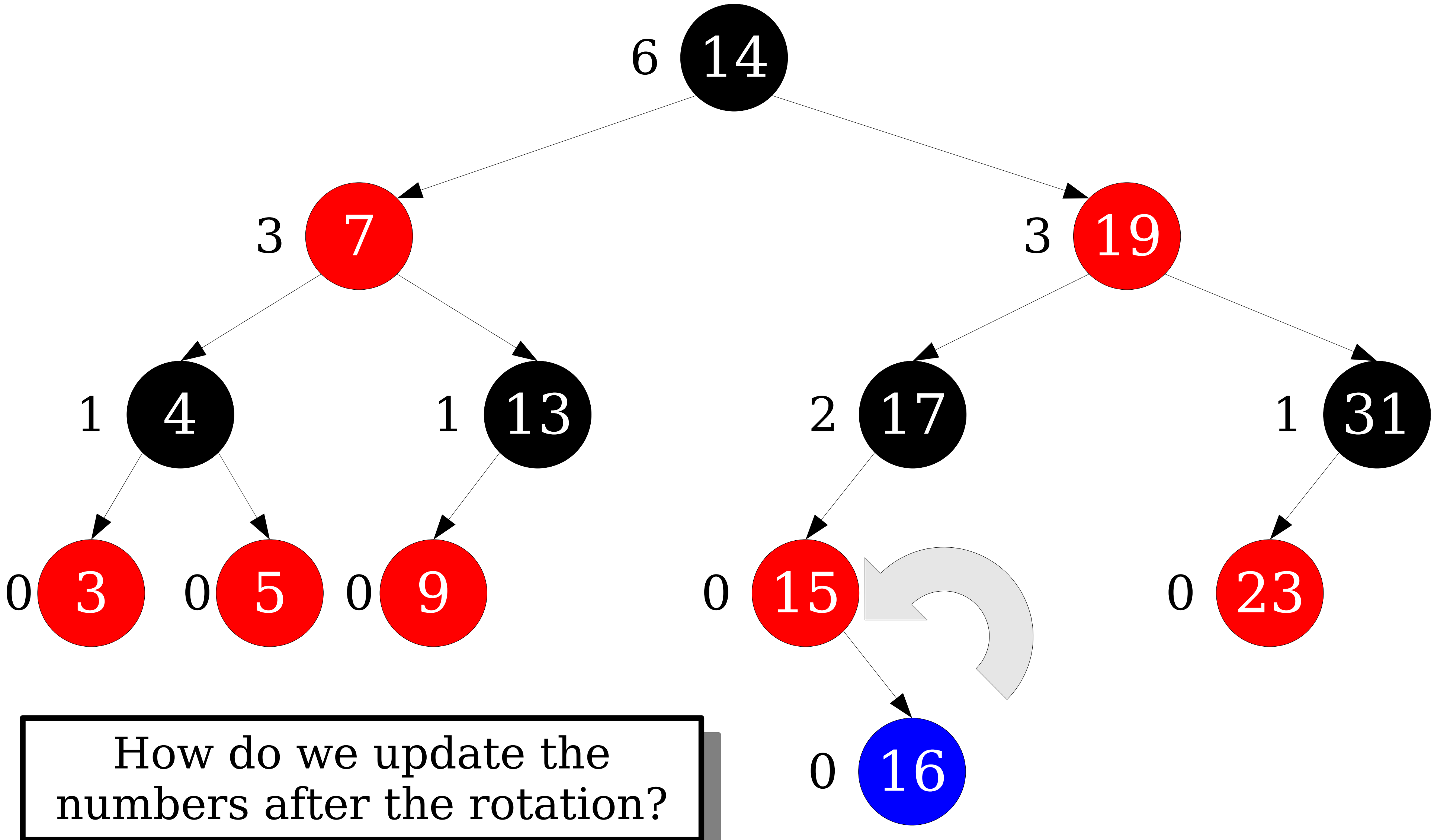


n_b
B
n_a
A
<A
>A
<B
>B
Rotate
Right
n_a
A
n_b - n_a - 1
B
<A
>A
<B
>B
?
B
n_a
A
<A
>A
<B
>B
Rotate
Left
n_a
A
n_b
B
<A
>A
<B
>B



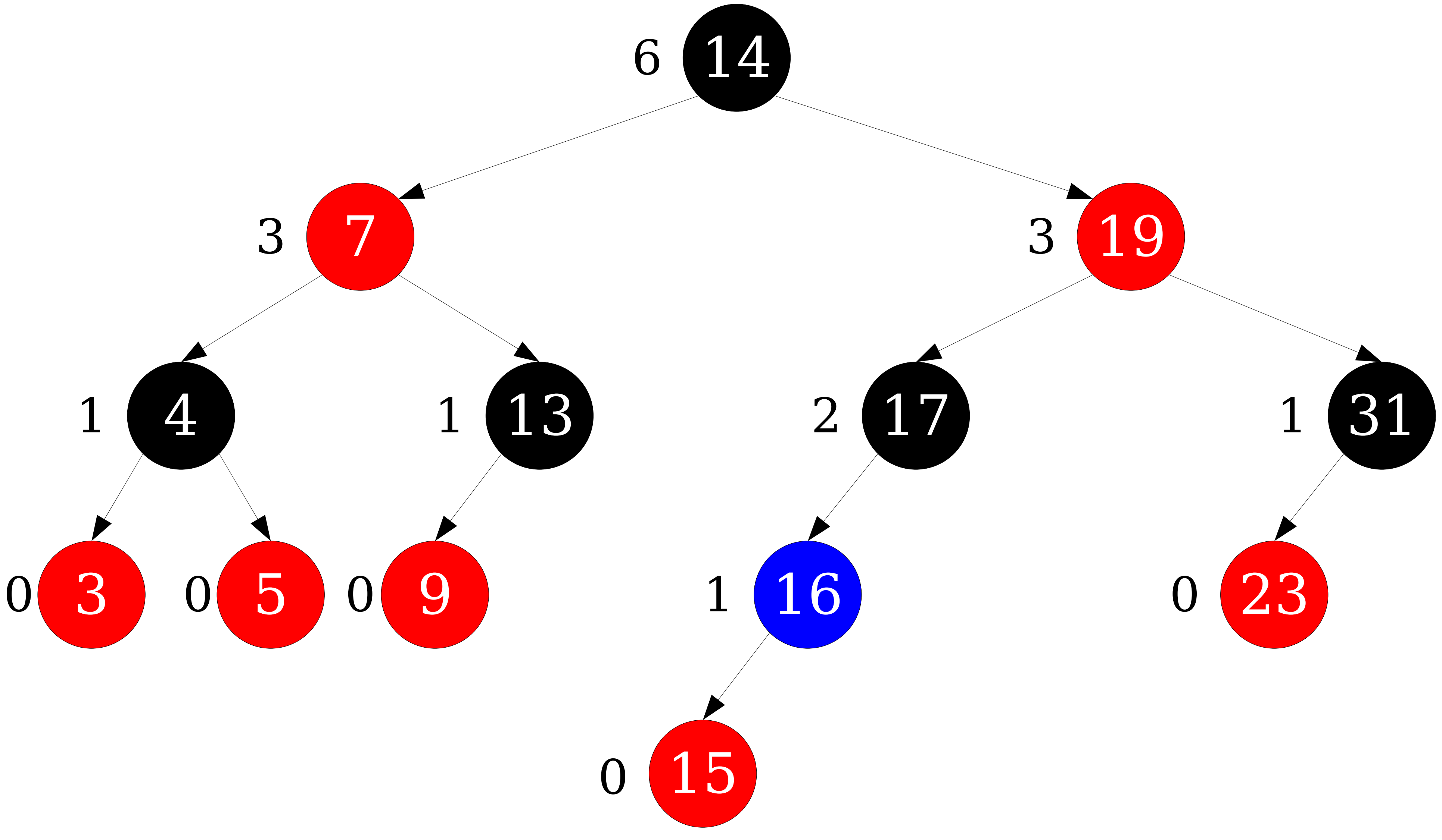
n_b
B
n_a
A
<A
>A
<B
>B
Rotate
Right
n_a
A
$n_b - n_a - 1$
B
<A
>A
<B
>B
$n_b + n_a + 1$
B
n_a
A
<A
>A
<B
>B
Rotate
Left
n_a
A
n_b
B
<A
>A
<B
>B

Dynamic Selection

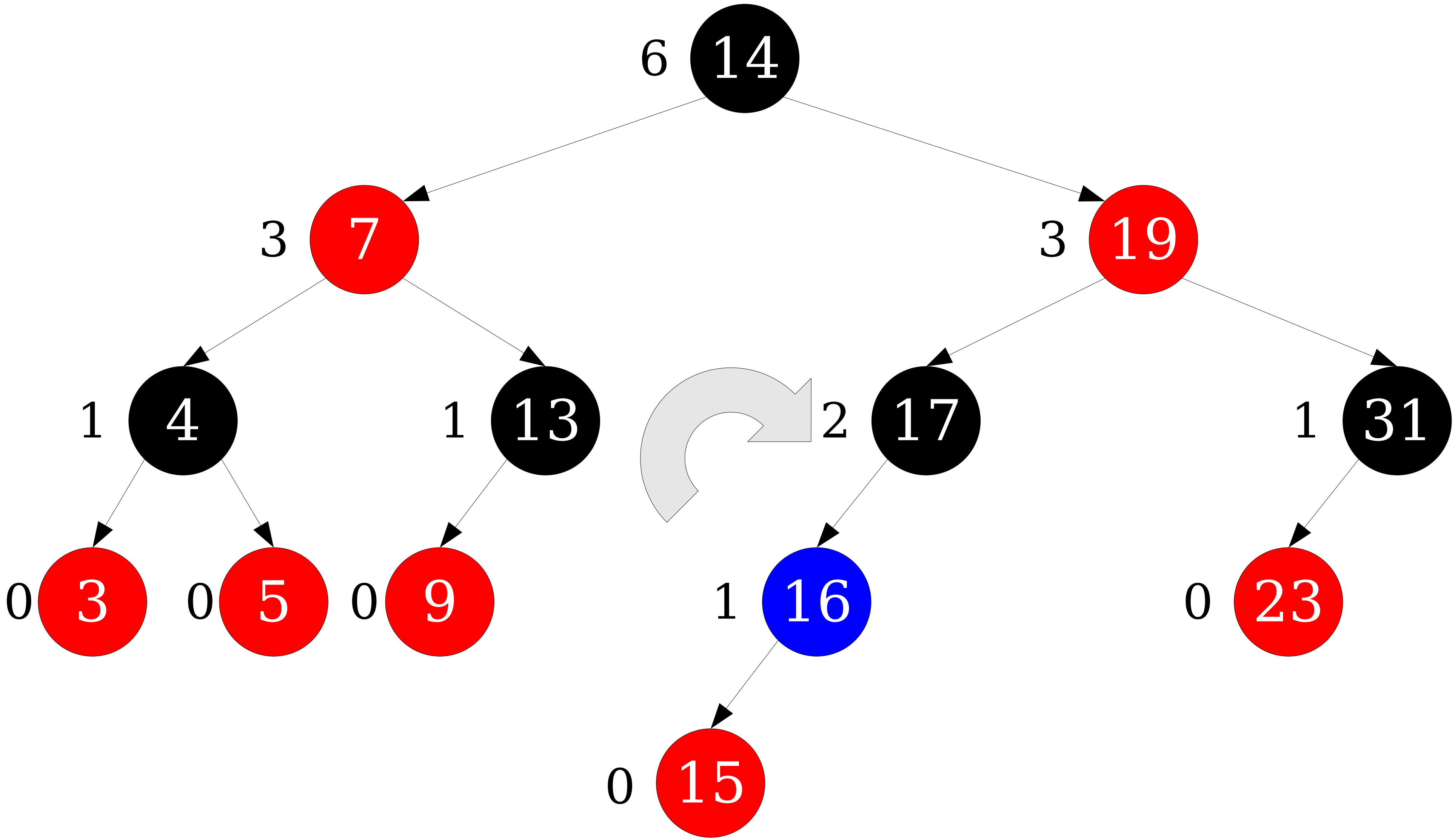


How do we update the numbers after the rotation?

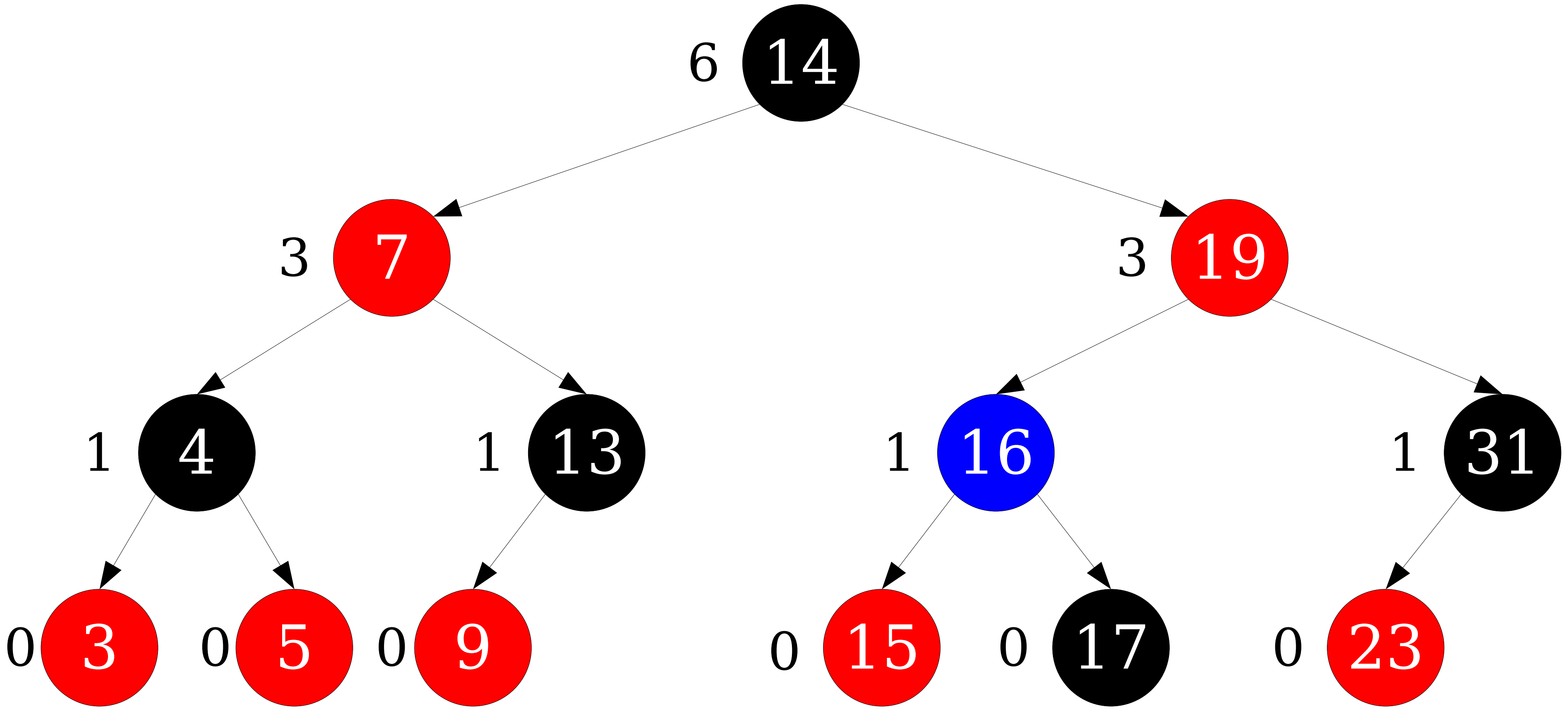
Dynamic Selection



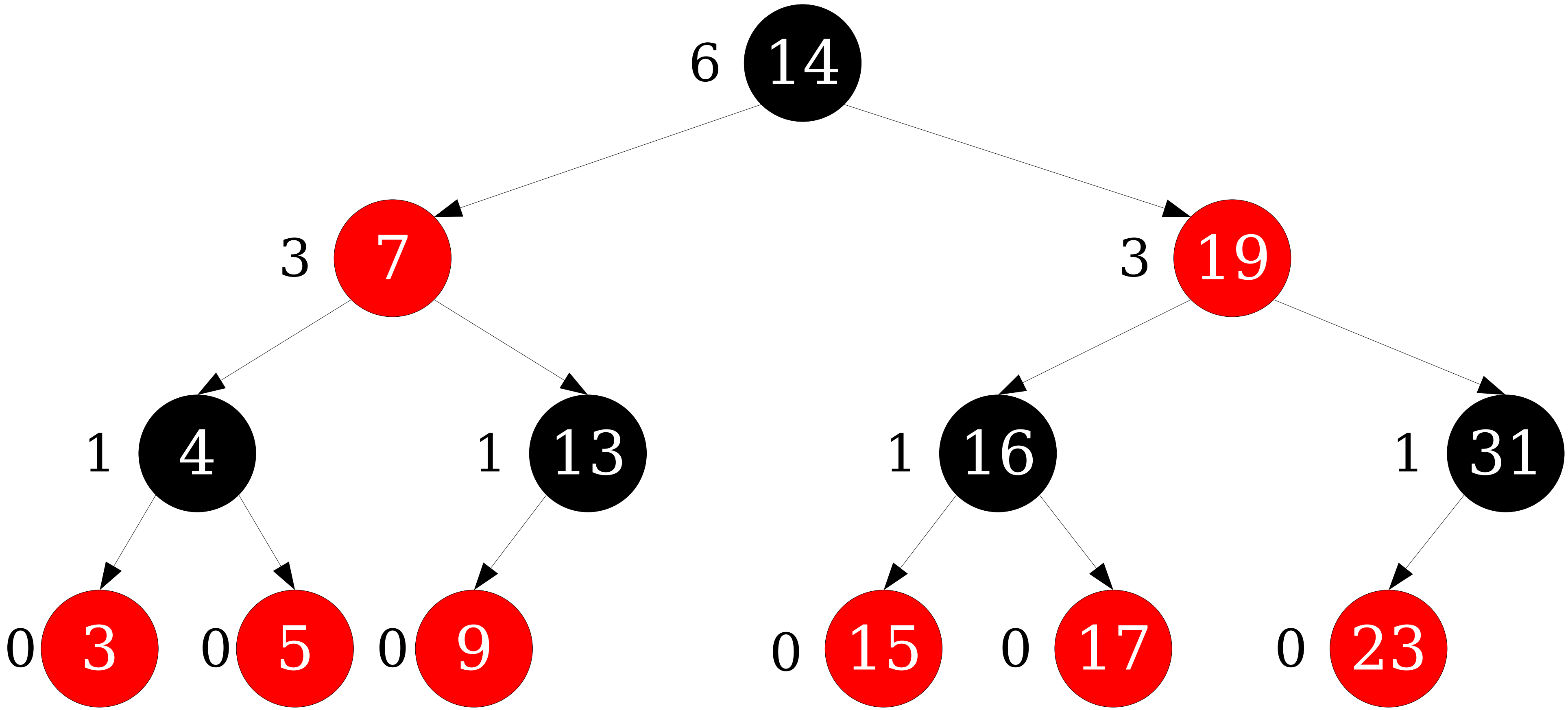
Dynamic Selection



Dynamic Selection



Dynamic Selection



Order Statistic Trees

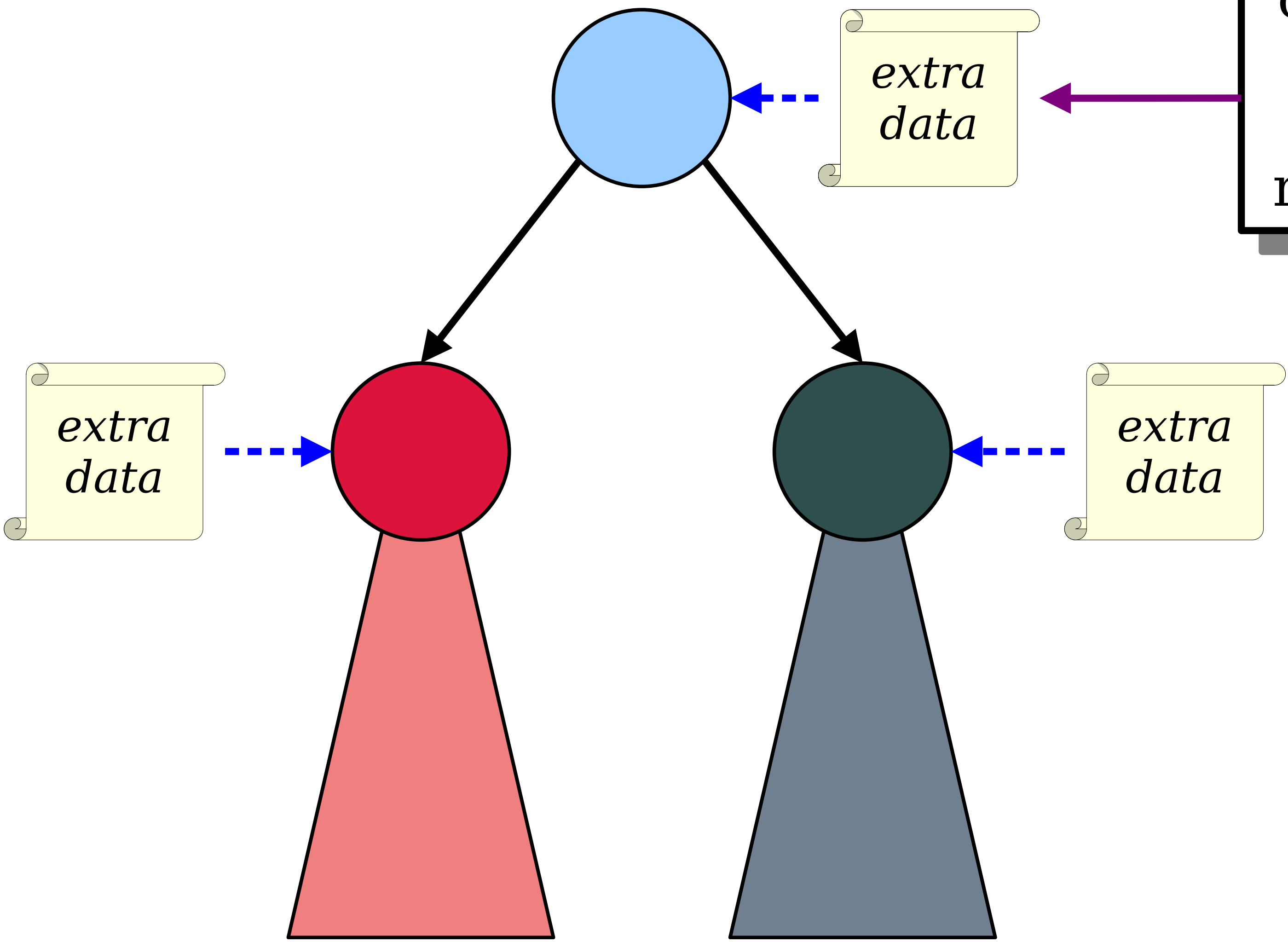
- This modified red/black tree is called an ***order statistic tree***.
  - Start with a red/black tree.
  - Tag each node with the number of nodes in its left subtree.
  - Use the preceding update rules to preserve values during rotations.
  - Propagate other changes up to the root of the tree.
- Only O(log n) values must be updated on an insertion or deletion and each can be updated in time O(1).
- They support all BST operations plus ***select*** (find kth order statistic) and ***rank*** (given a key, report its order statistic) in time O(log n).

Generalizing our Idea

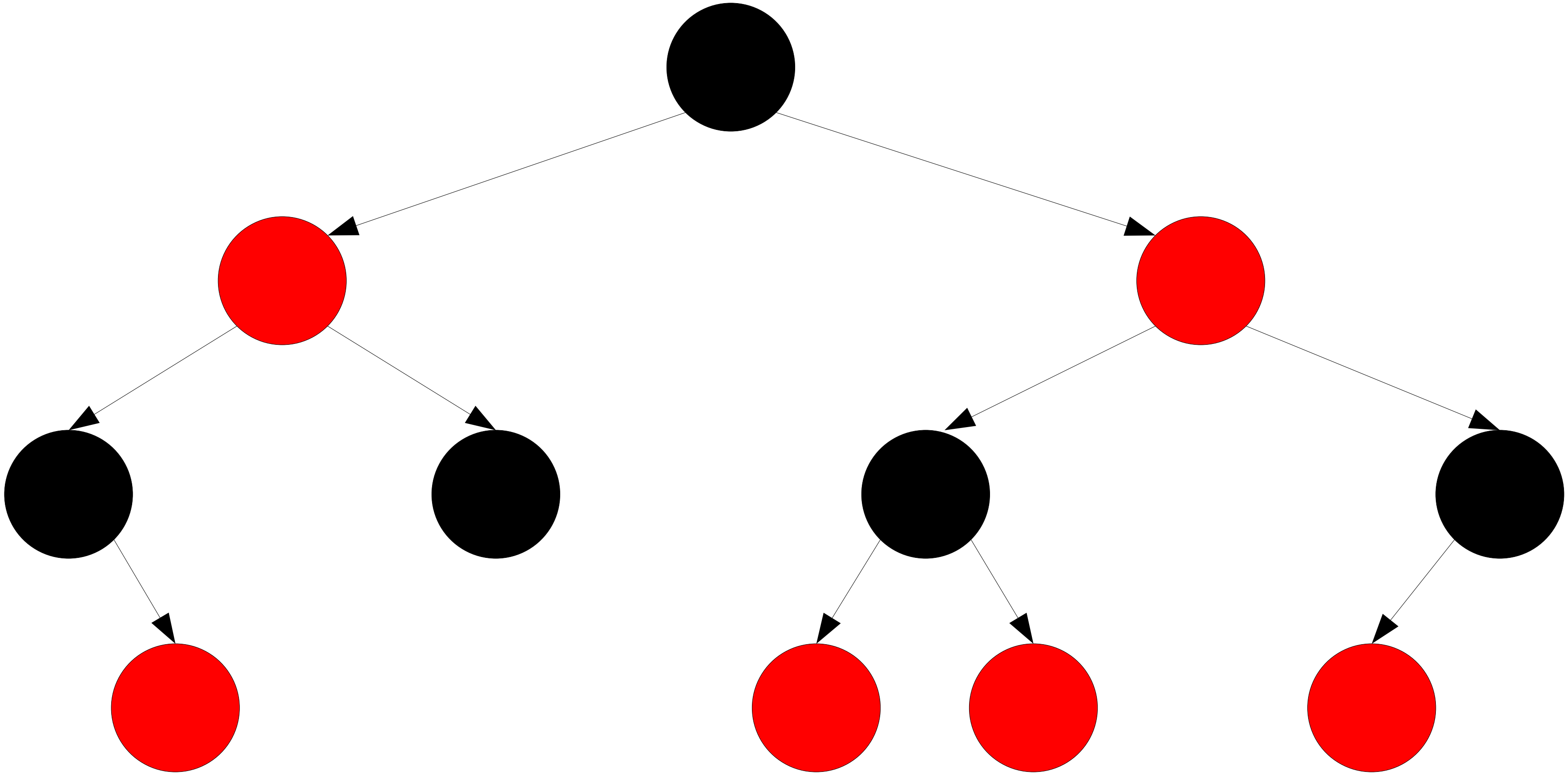
What Makes This Work?

- We started with a vanilla red/black tree.
- We ***augmented*** the tree by storing some extra information in each node.
- On an insertion or deletion, that information only needed to be updated
  - along the access path, and
  - in nodes undergoing rotation.
- Each update takes time O(1).
- At most O(log n) nodes need updates, so the extra cost of maintaining the information is O(log n) per insertion or deletion.

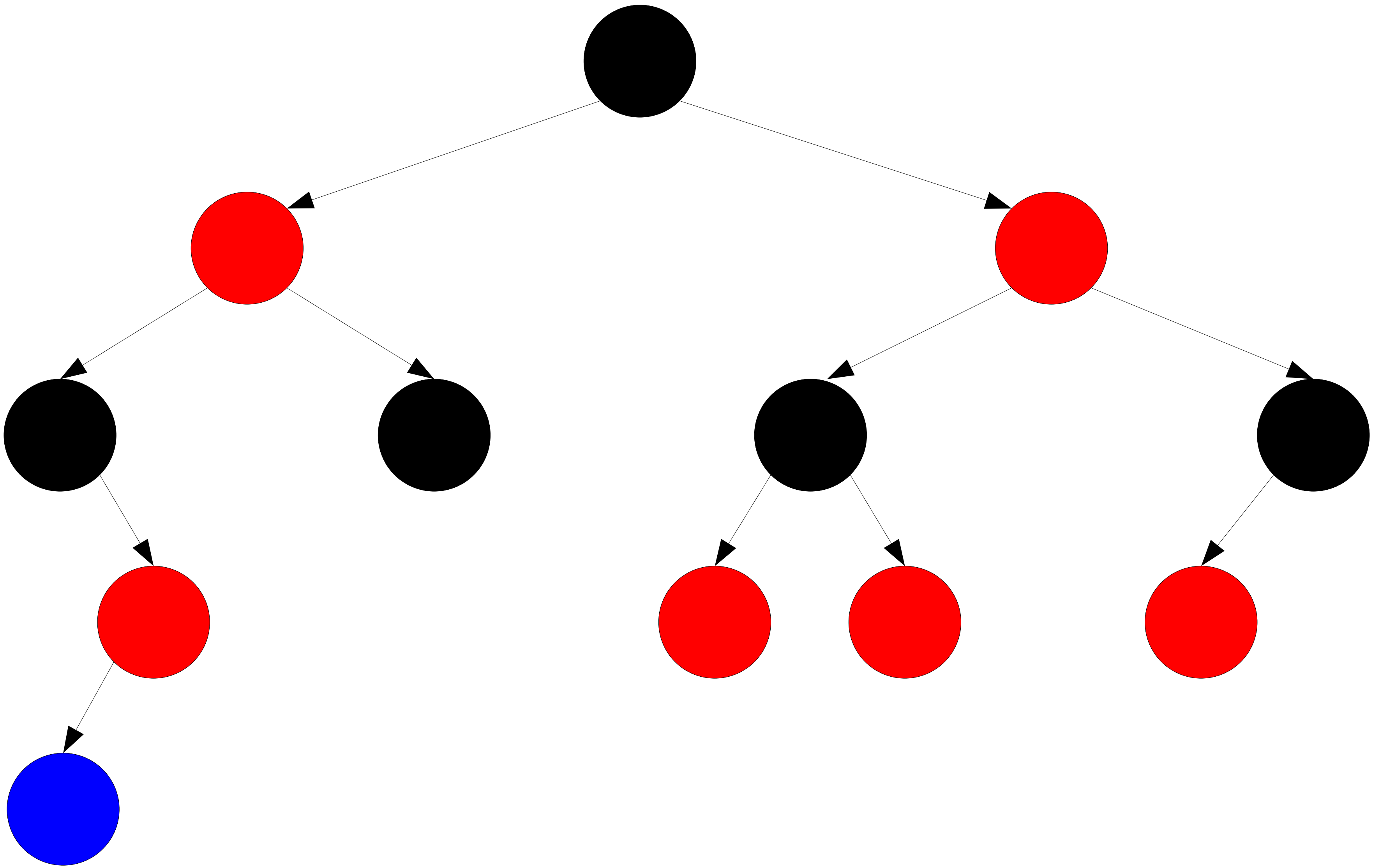
Generalizing This Idea



Assumption: We can determine what goes here in time O(1) based on the node's two children.



Imagine we cache some value in each node that can be computed just from (1) the node itself and (2) its children's values.



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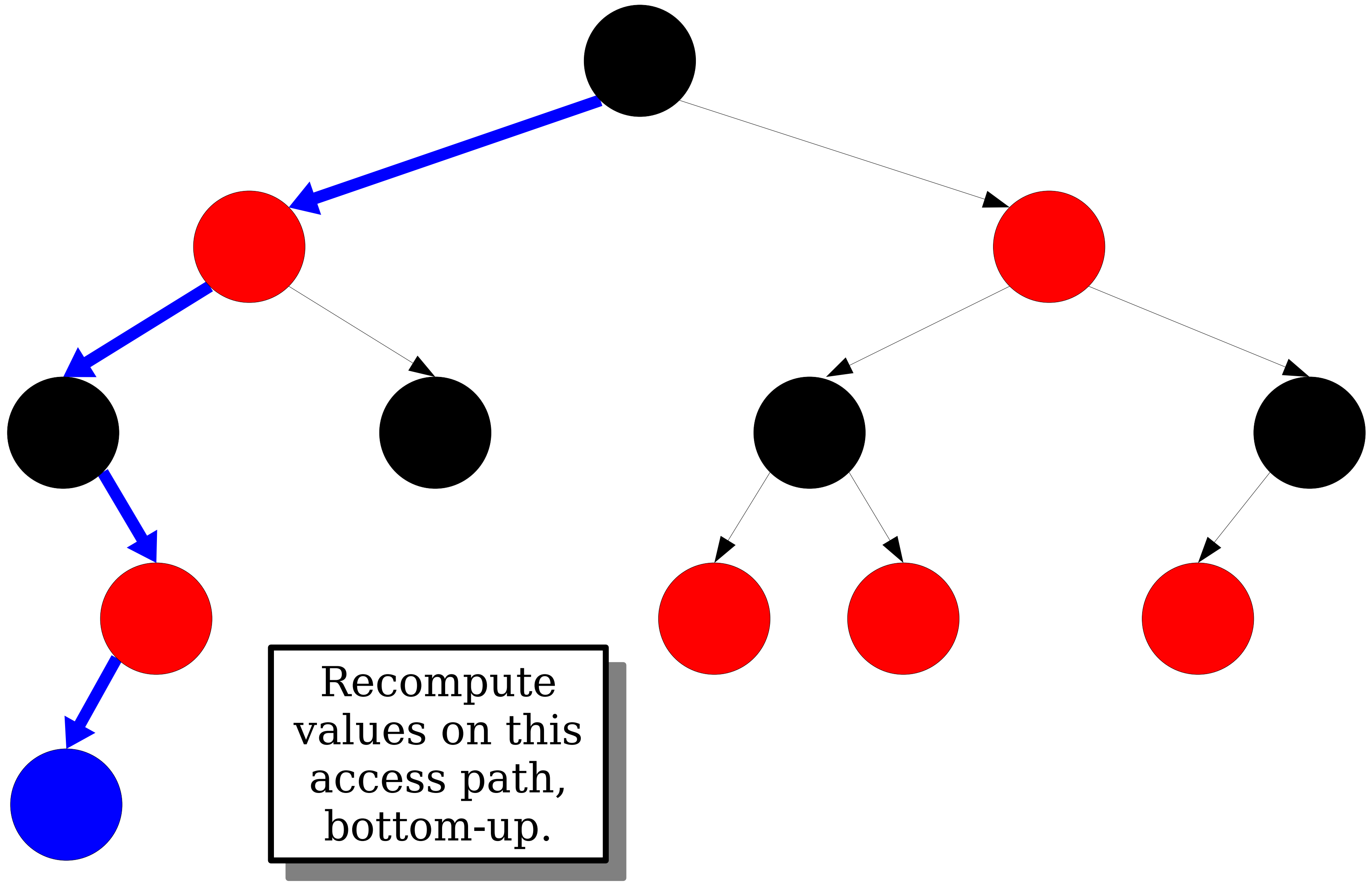

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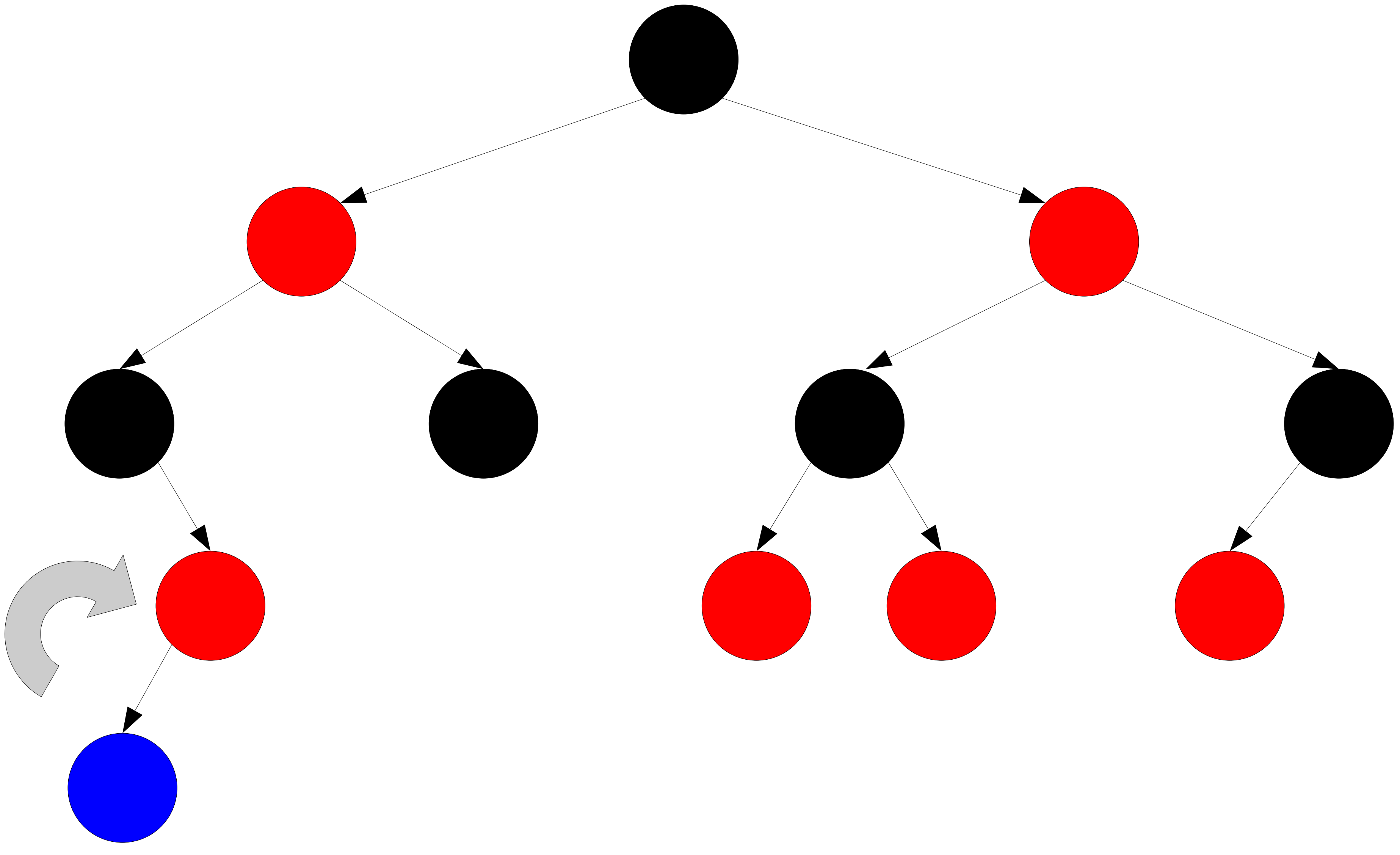

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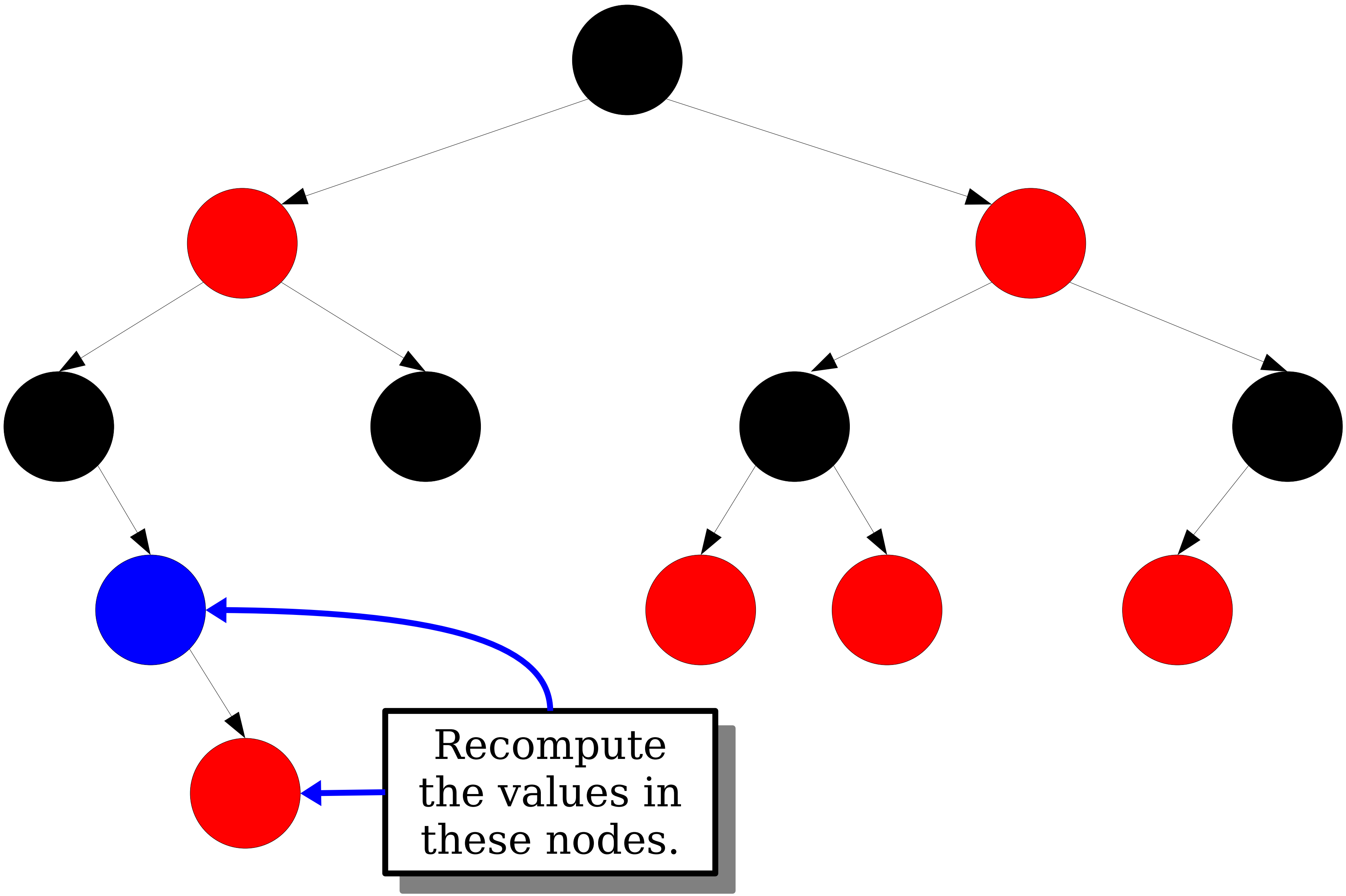

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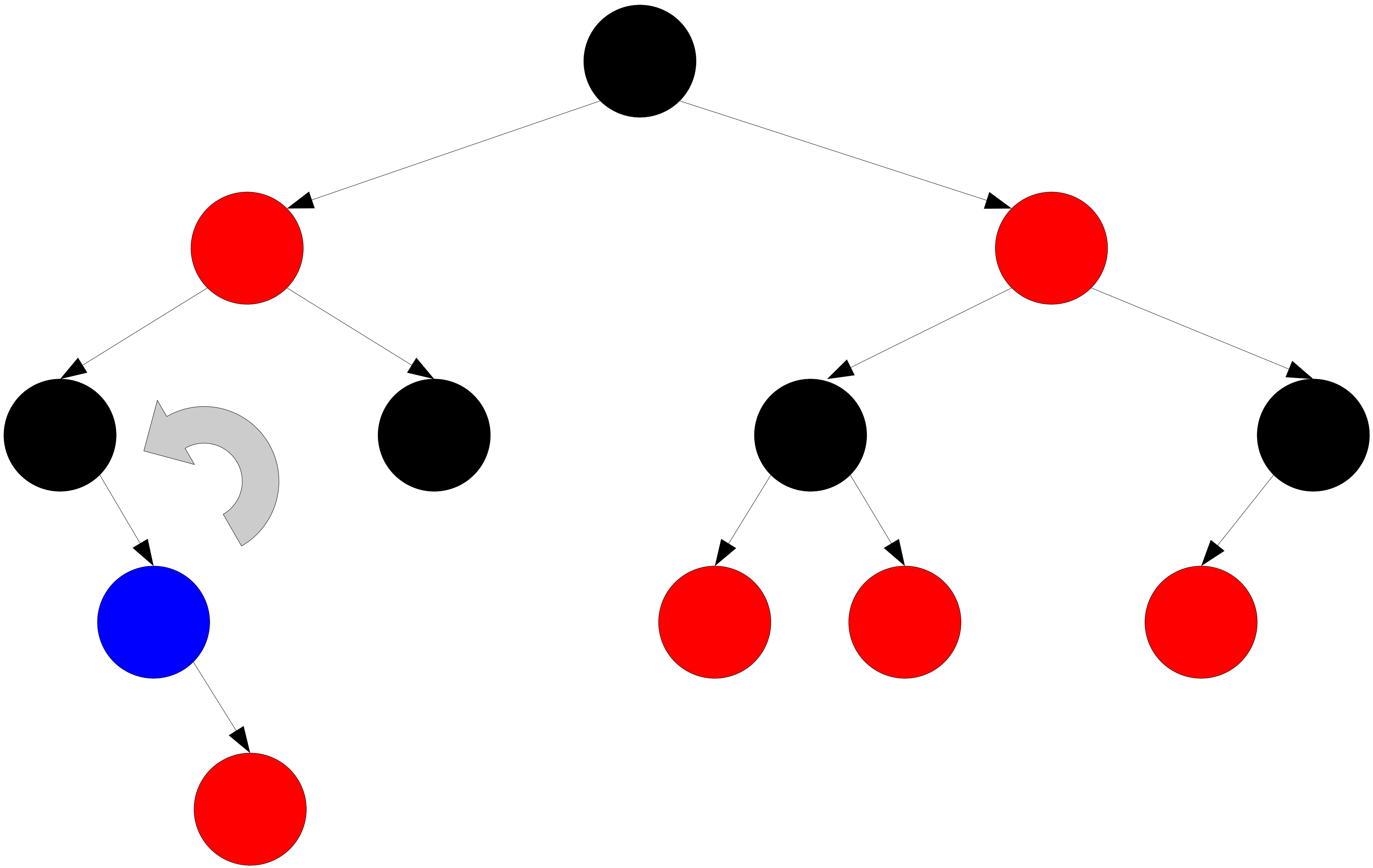

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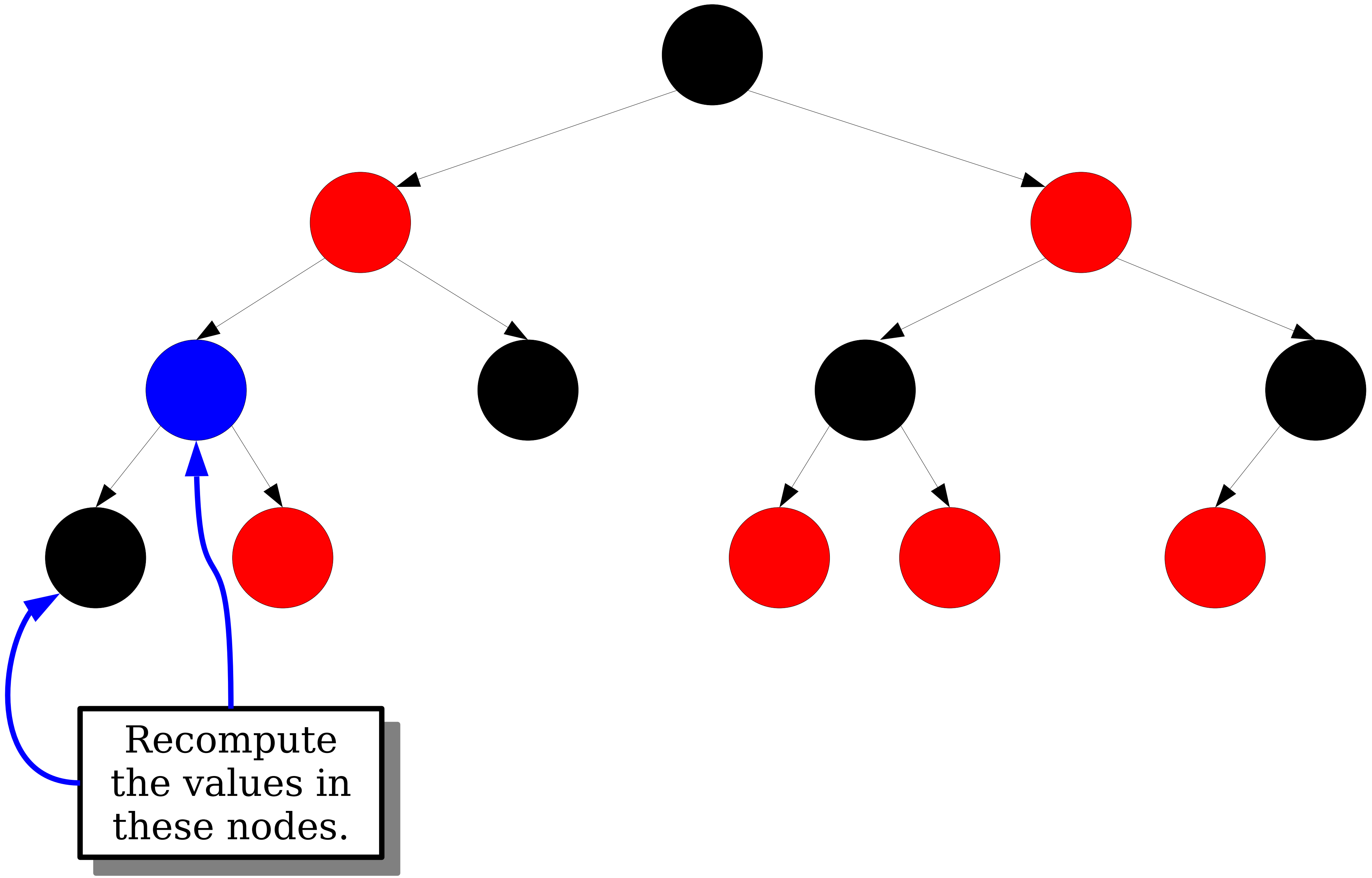

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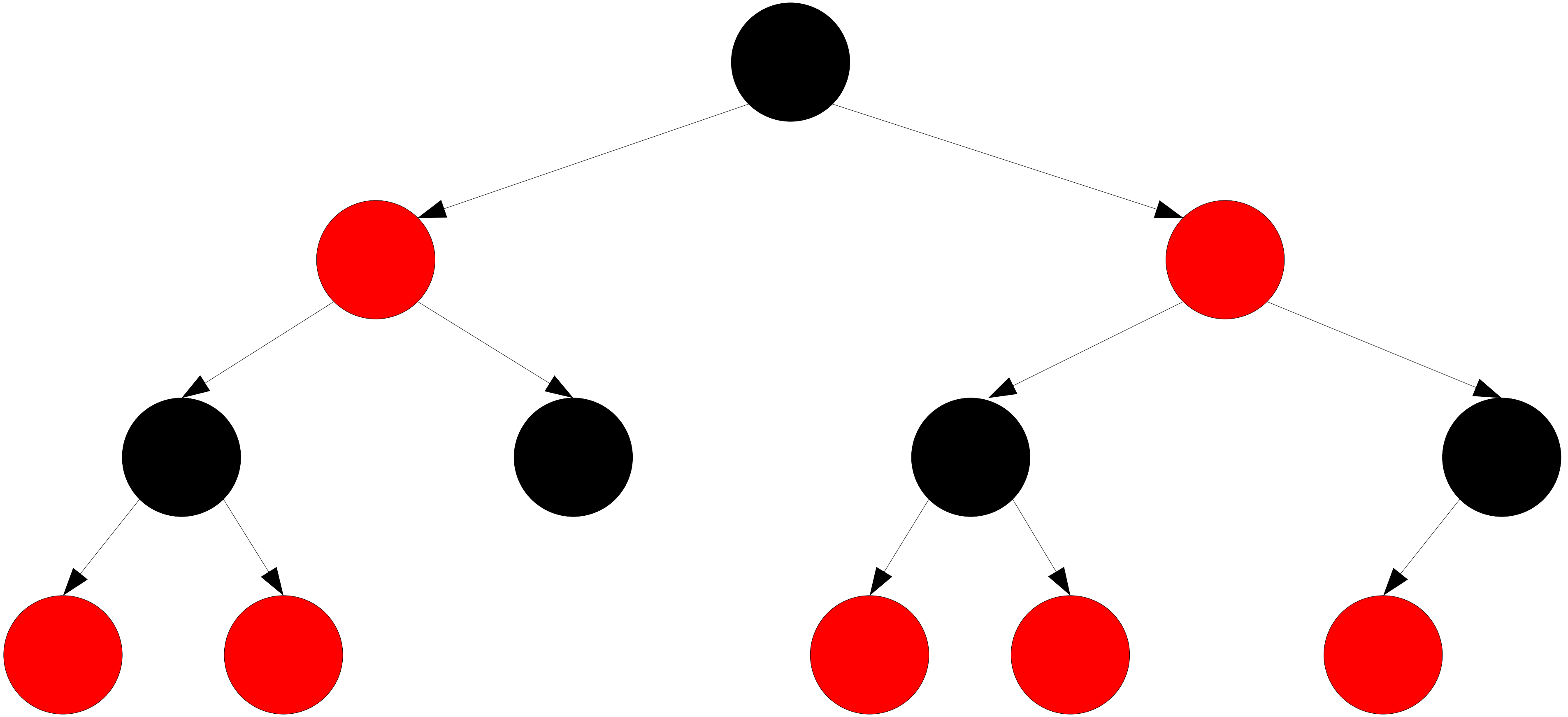

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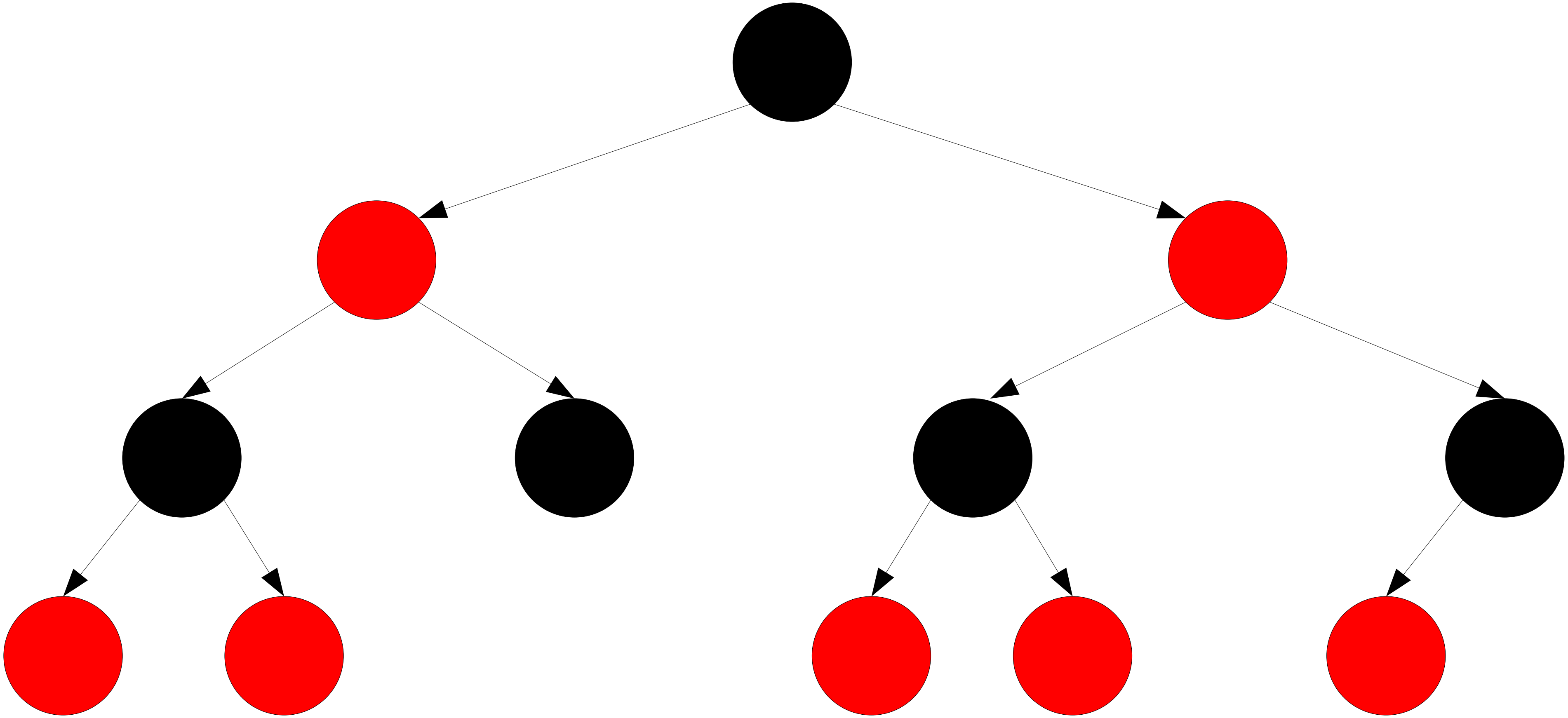


Theorem: Suppose we want to cache some computed value in each node of a red/black tree. Provided that the value can be recomputed purely from the node's value and from it's children's values, and provided that each value can be computed in time O(1), then these values can be cached in each node with insertions, lookups, and deletions still taking time O(log *n*).

Example: ***Hierarchical Clustering***

1D Hierarchical Clustering

20	**42**	**44**	**60**	**66**	**71**	**86**	**92**	**100**

1D Hierarchical Clustering

20	**42**	**44**	**60**	**66**	**71**	**86**	**92**	**100**

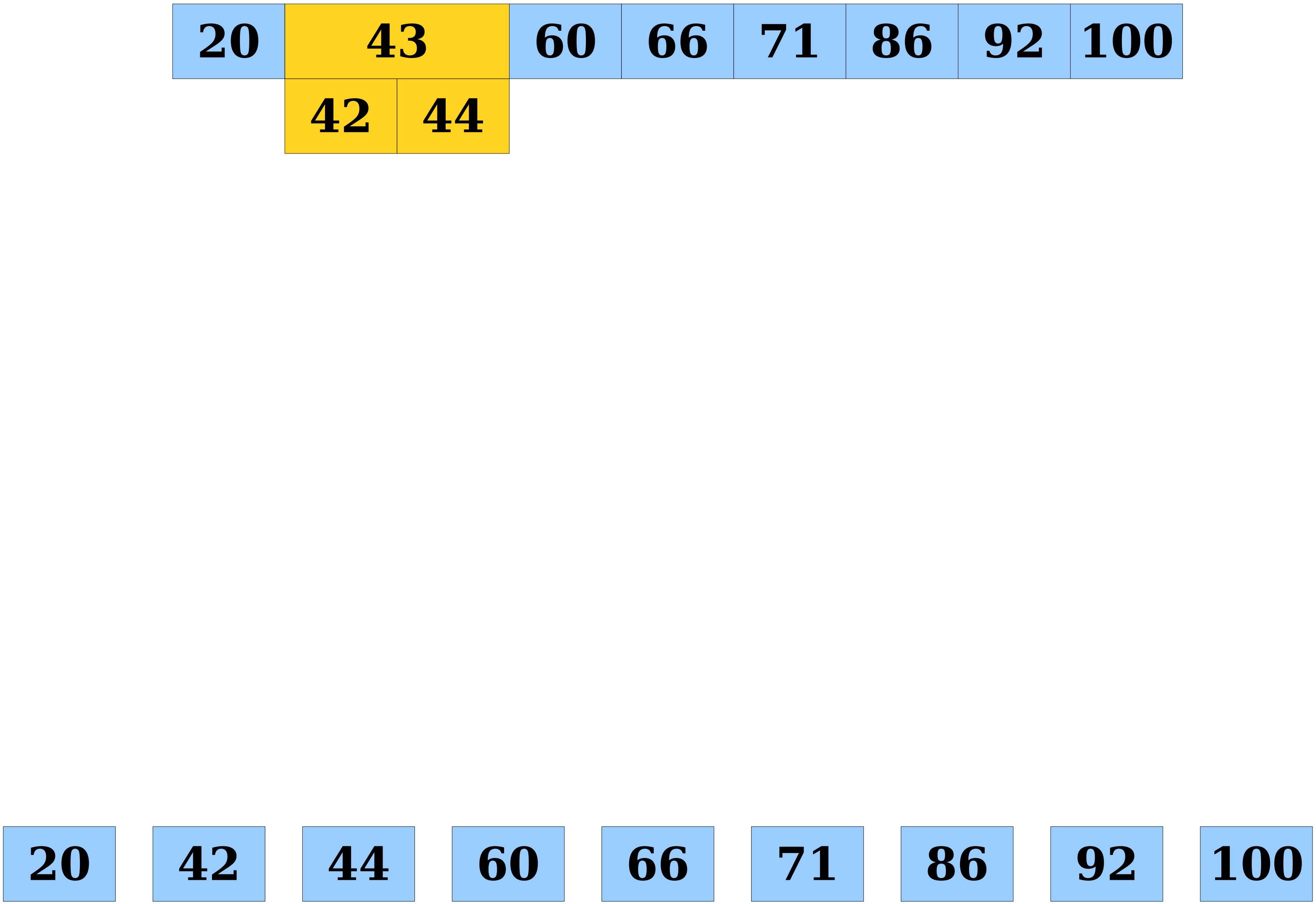
20	**42**	**44**	**60**	**66**	**71**	**86**	**92**	**100**

1D Hierarchical Clustering

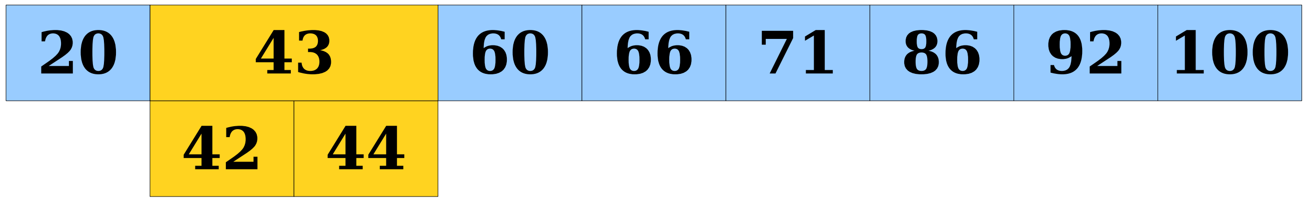
20	**42**	**44**	**60**	**66**	**71**	**86**	**92**	**100**

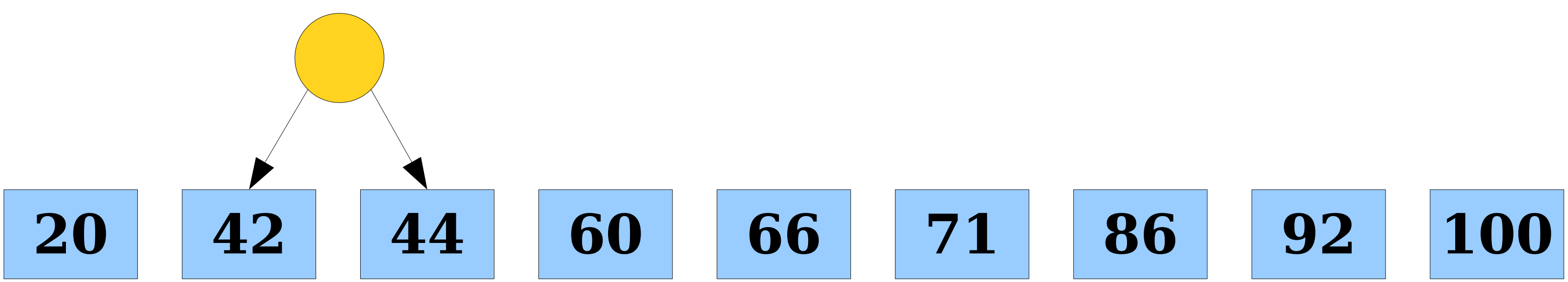
20	**42**	**44**	**60**	**66**	**71**	**86**	**92**	**100**

1D Hierarchical Clustering

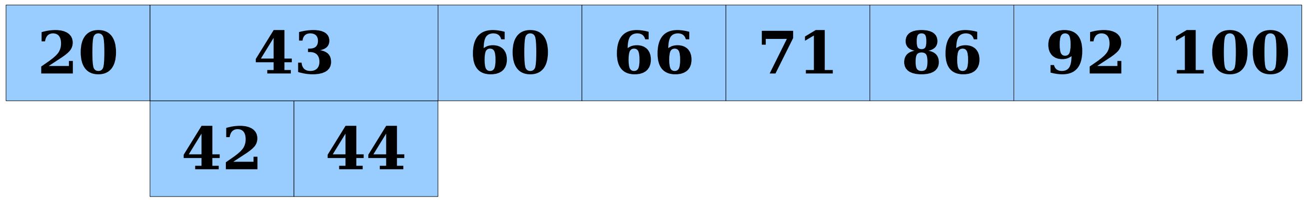


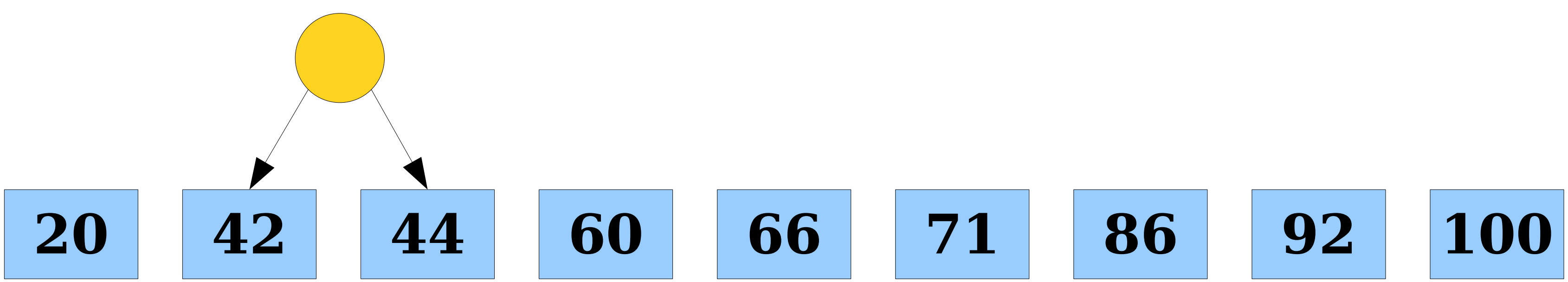
1D Hierarchical Clustering



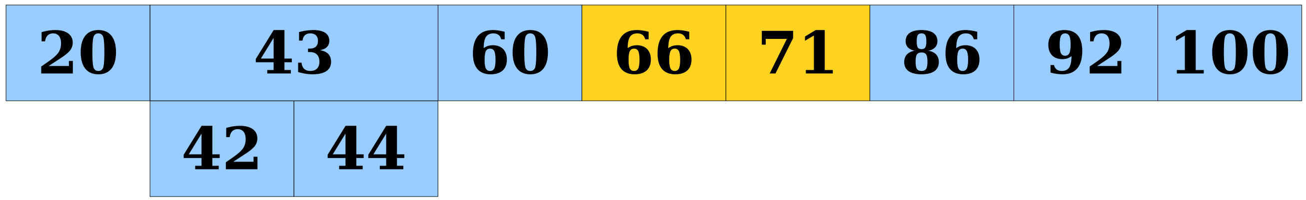


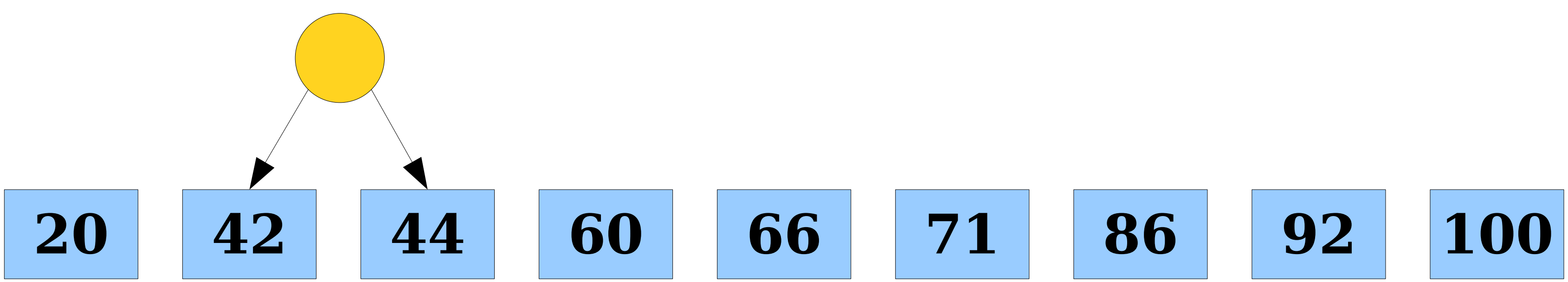
1D Hierarchical Clustering



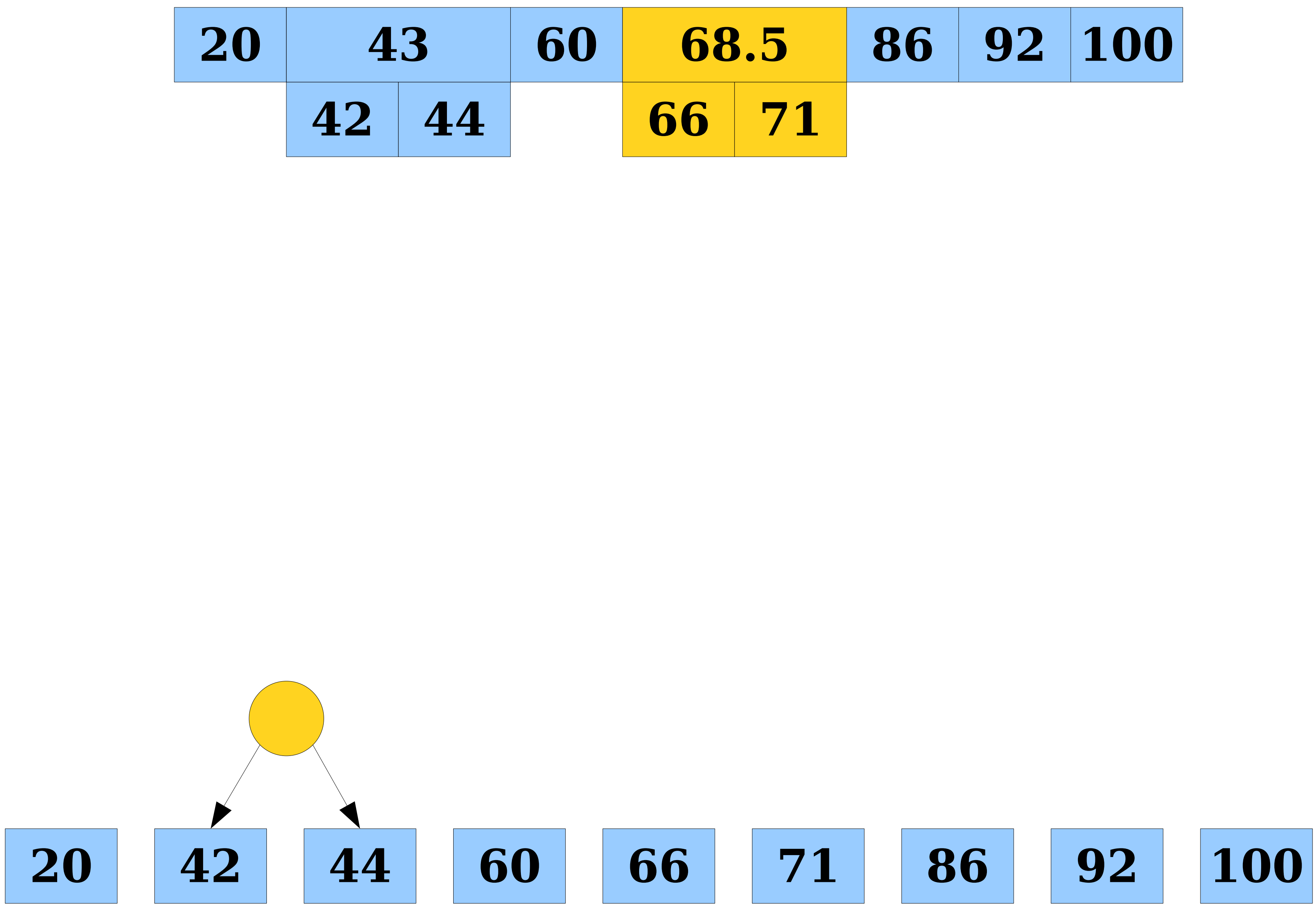


1D Hierarchical Clustering

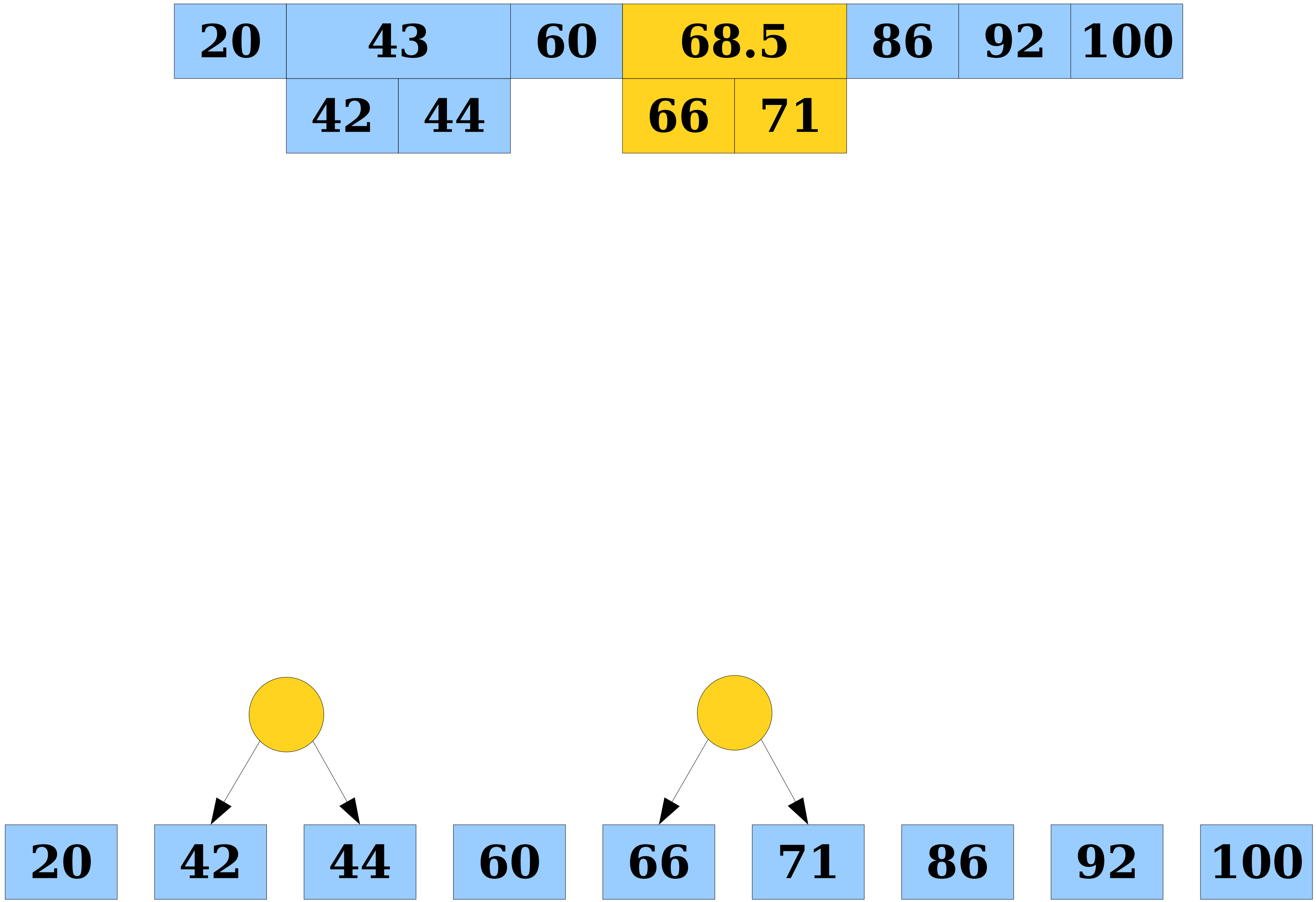




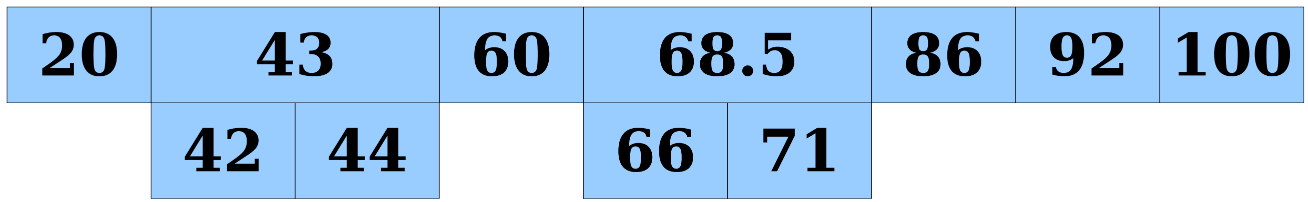
1D Hierarchical Clustering

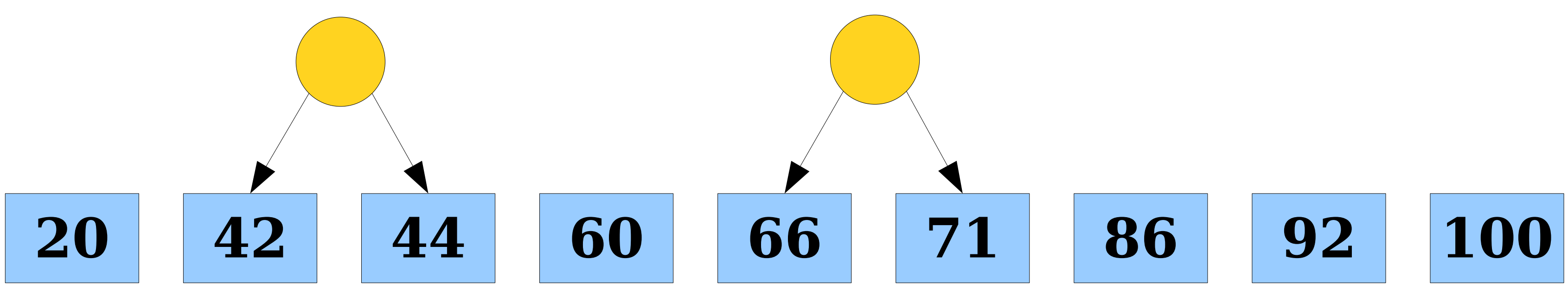


1D Hierarchical Clustering

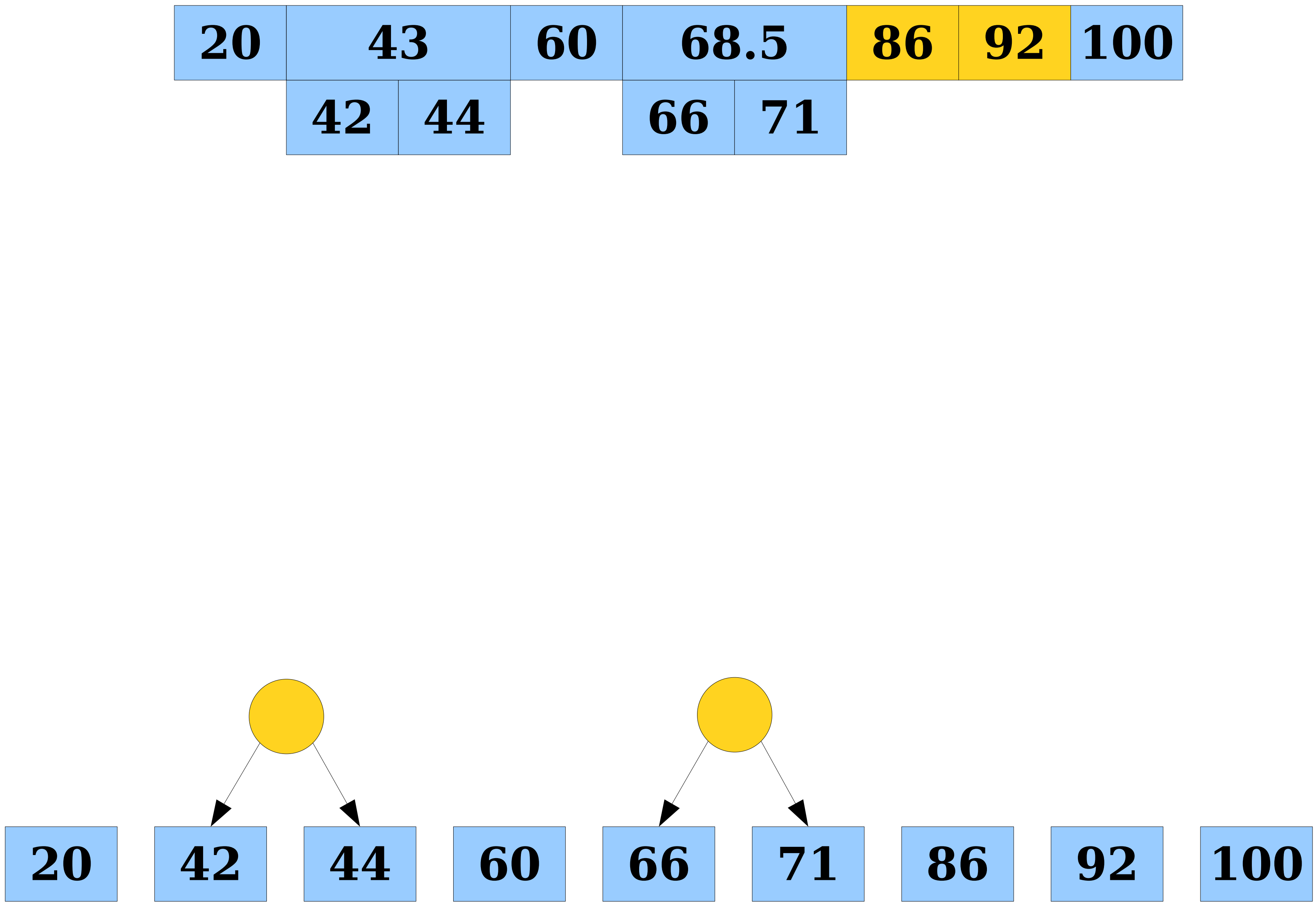


1D Hierarchical Clustering

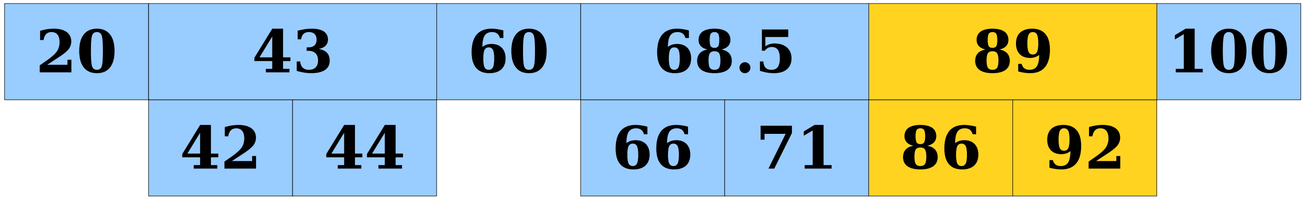


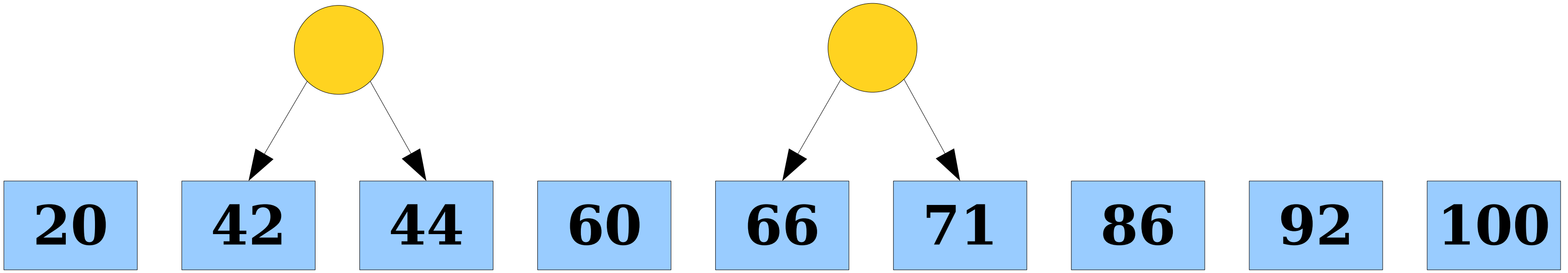


1D Hierarchical Clustering

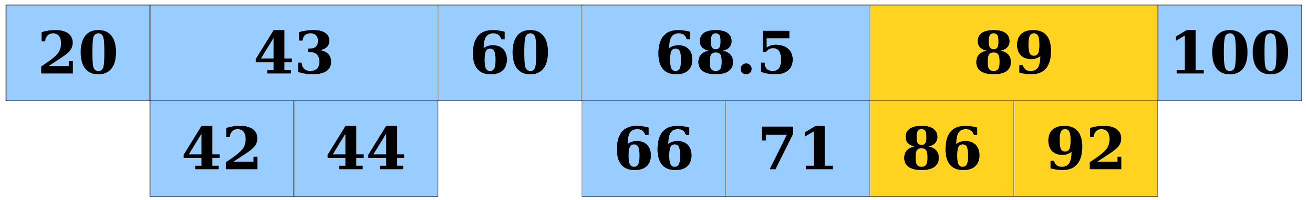


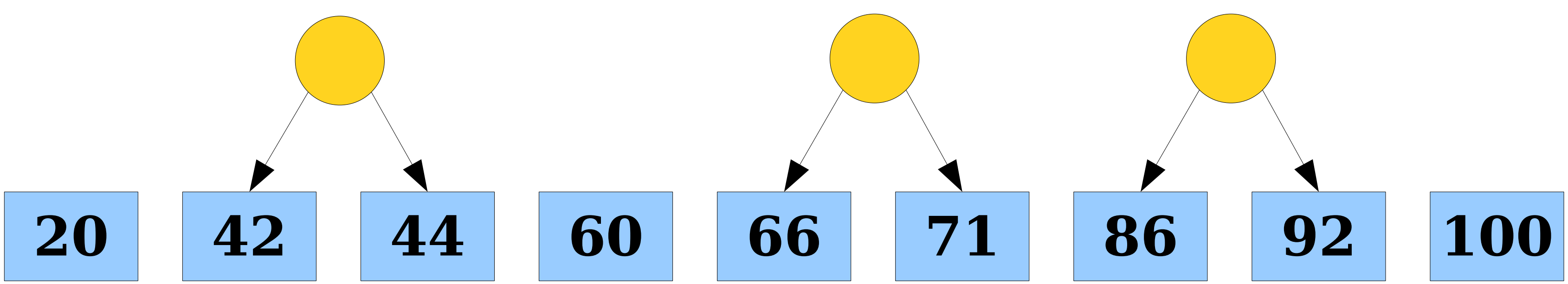
1D Hierarchical Clustering





1D Hierarchical Clustering





1D Hierarchical Clustering

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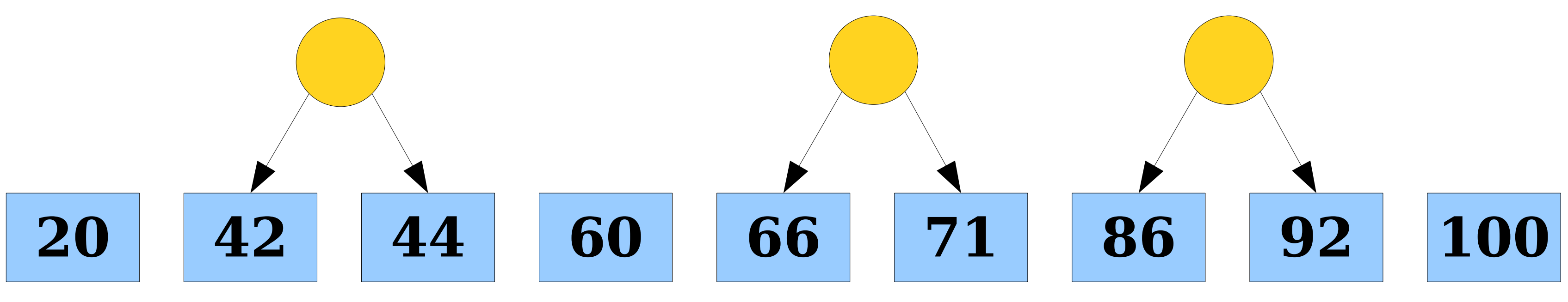

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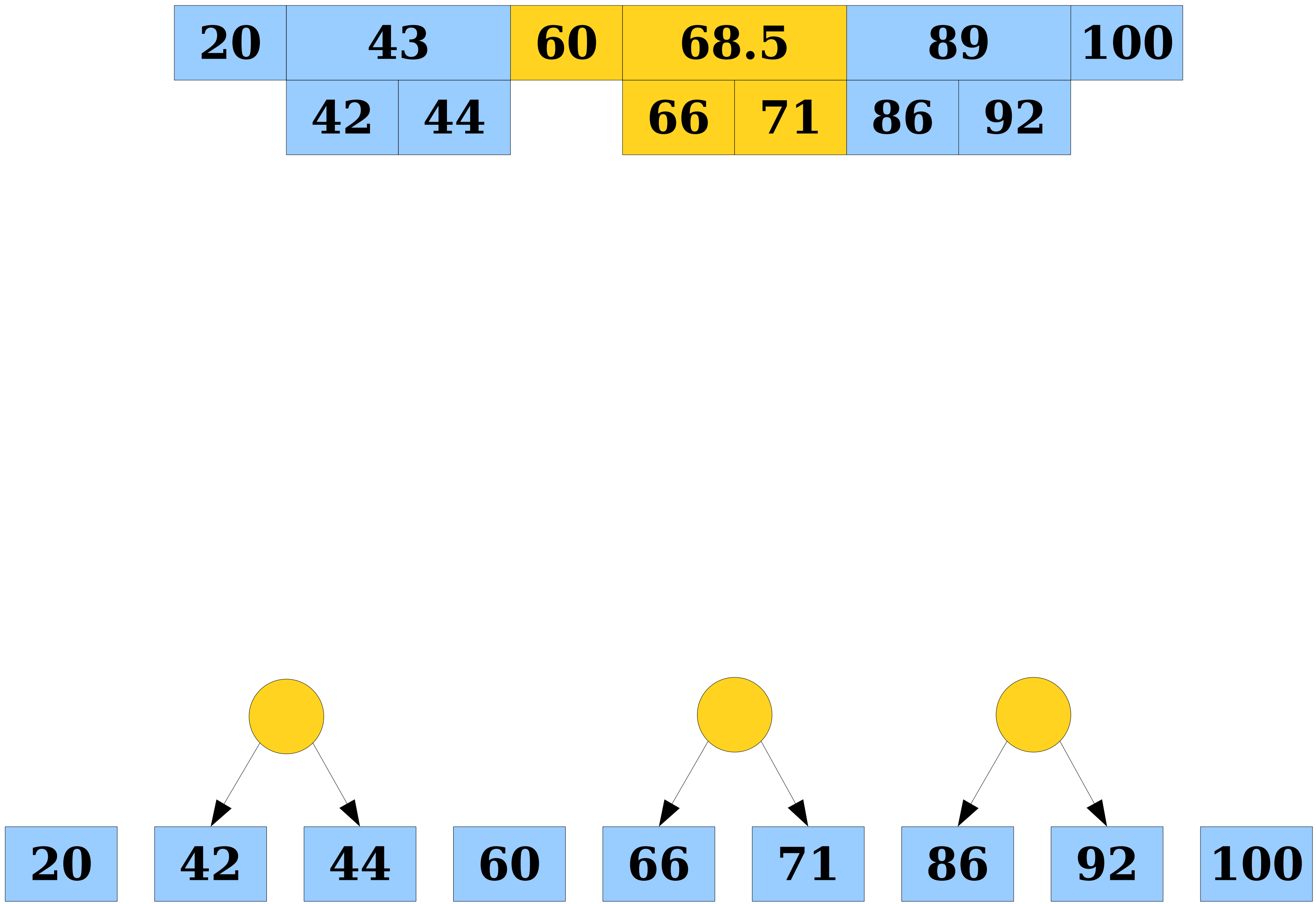

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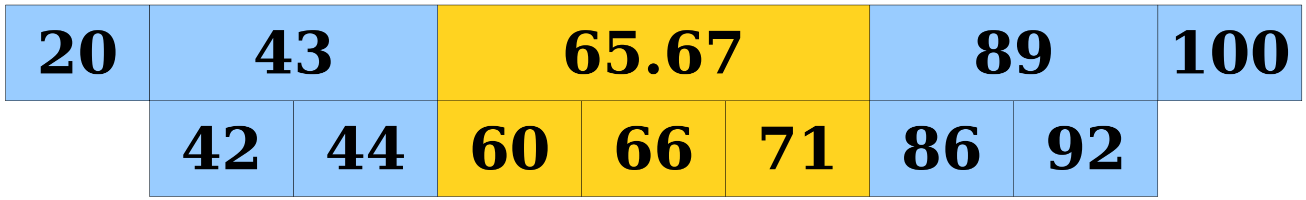

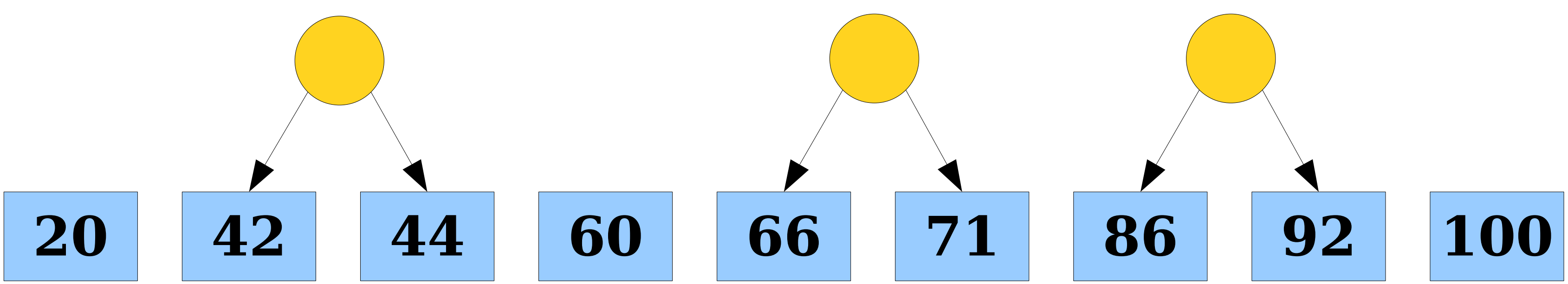

1D Hierarchical Clustering

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<tr><td></td><td>42</td><td>44</td><td>60</td><td>66</td><td>71</td><td>86</td><td>92</td><td></td></tr>
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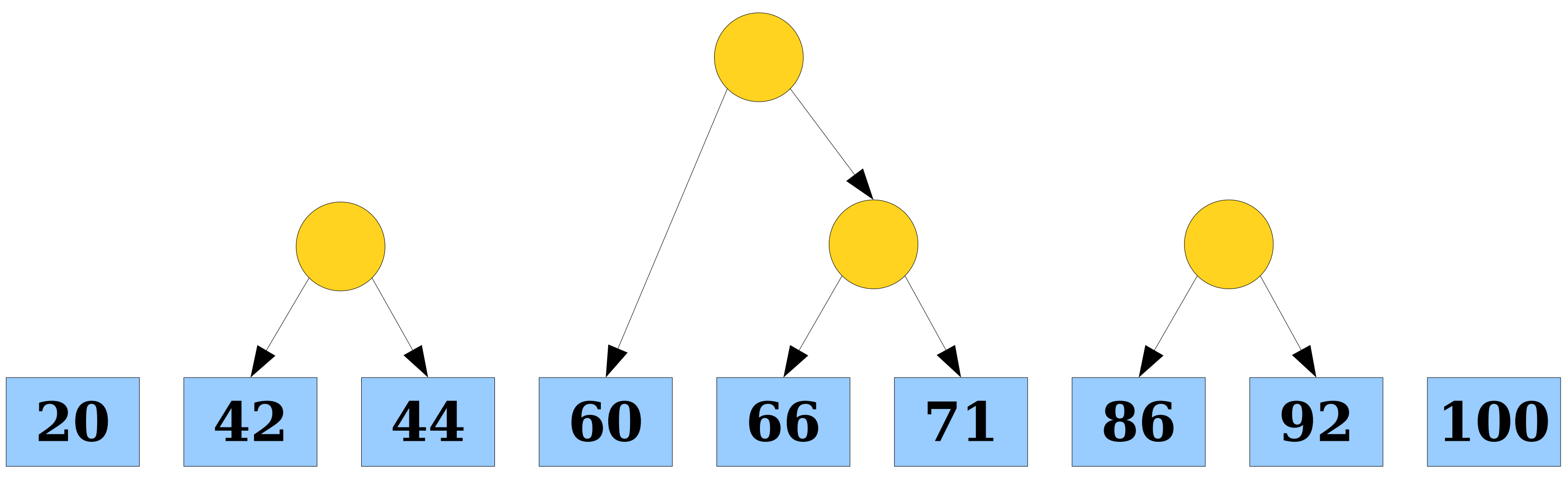
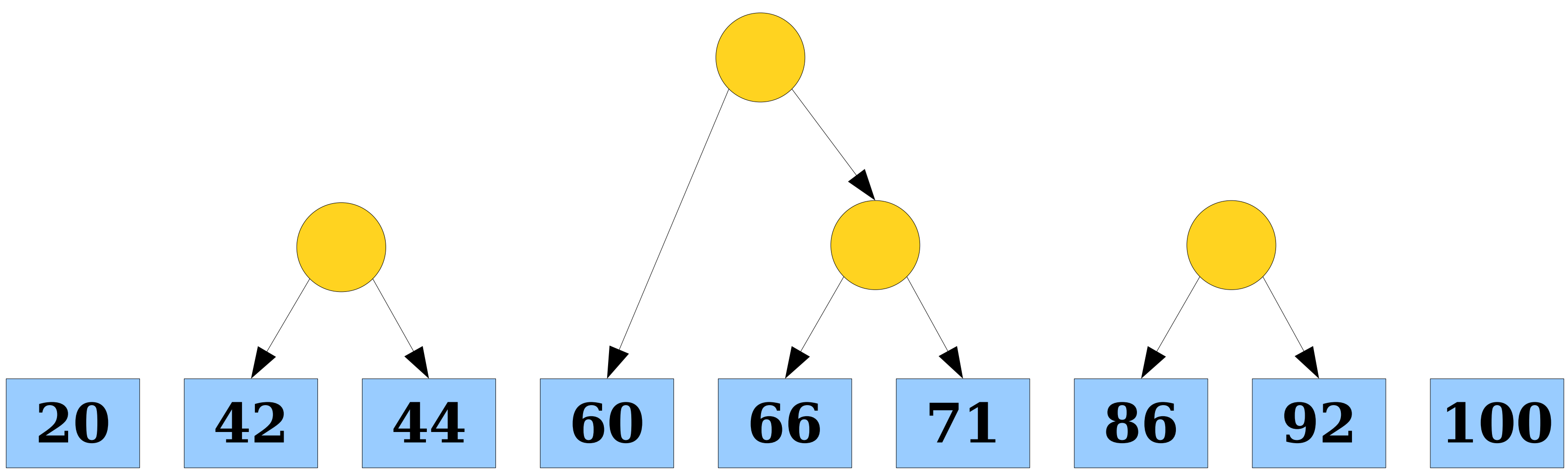
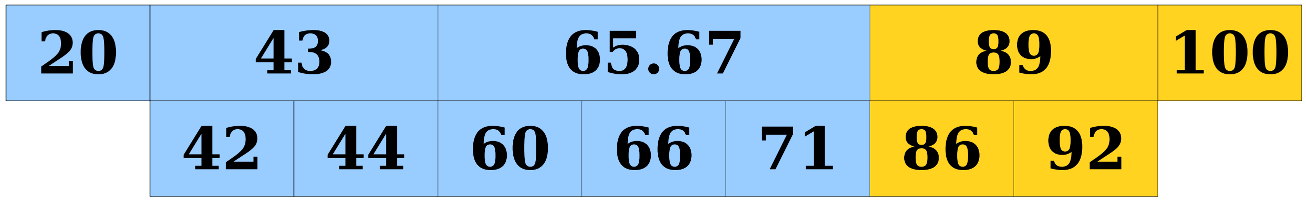


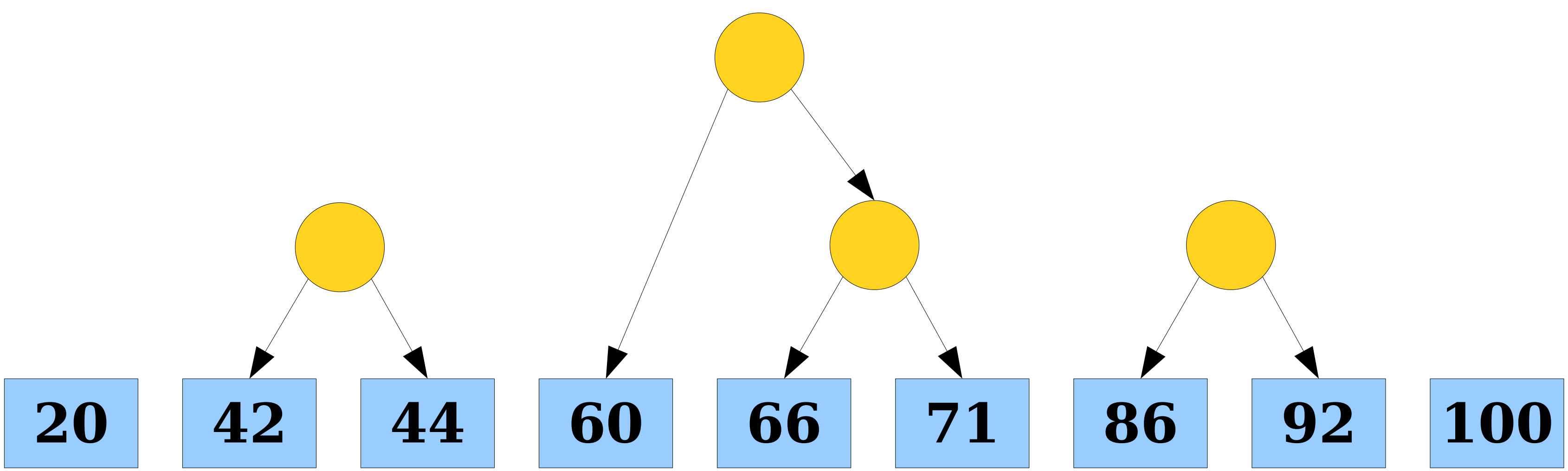
1D Hierarchical Clustering

20	43		65.67			89		100
	42	44	60	66	71	86	92	



1D Hierarchical Clustering





1D Hierarchical Clustering

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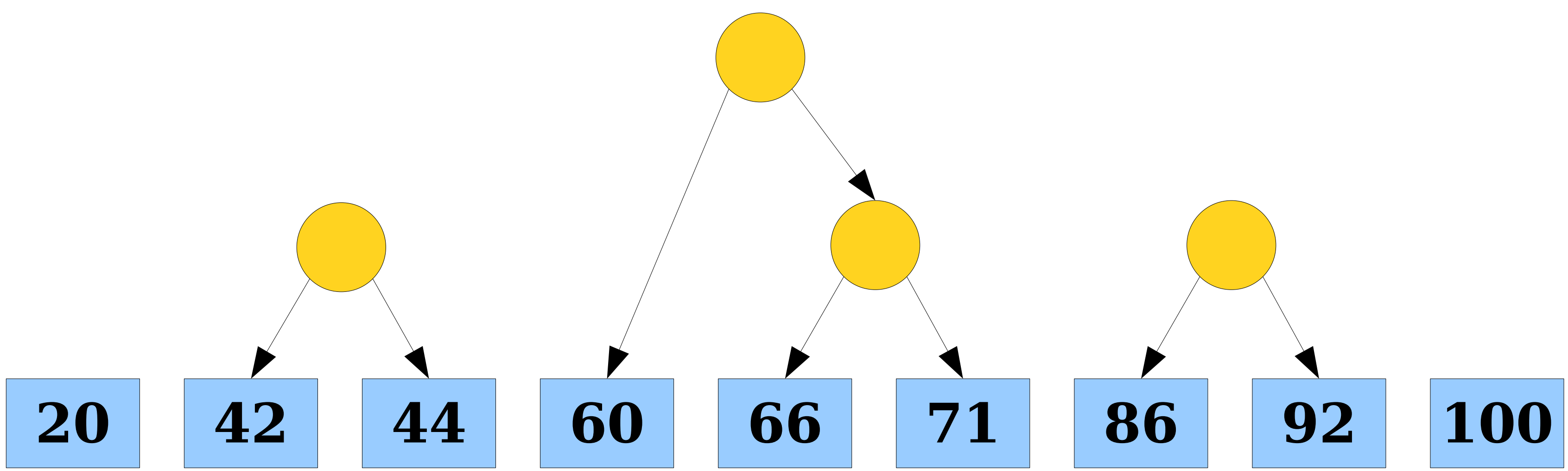
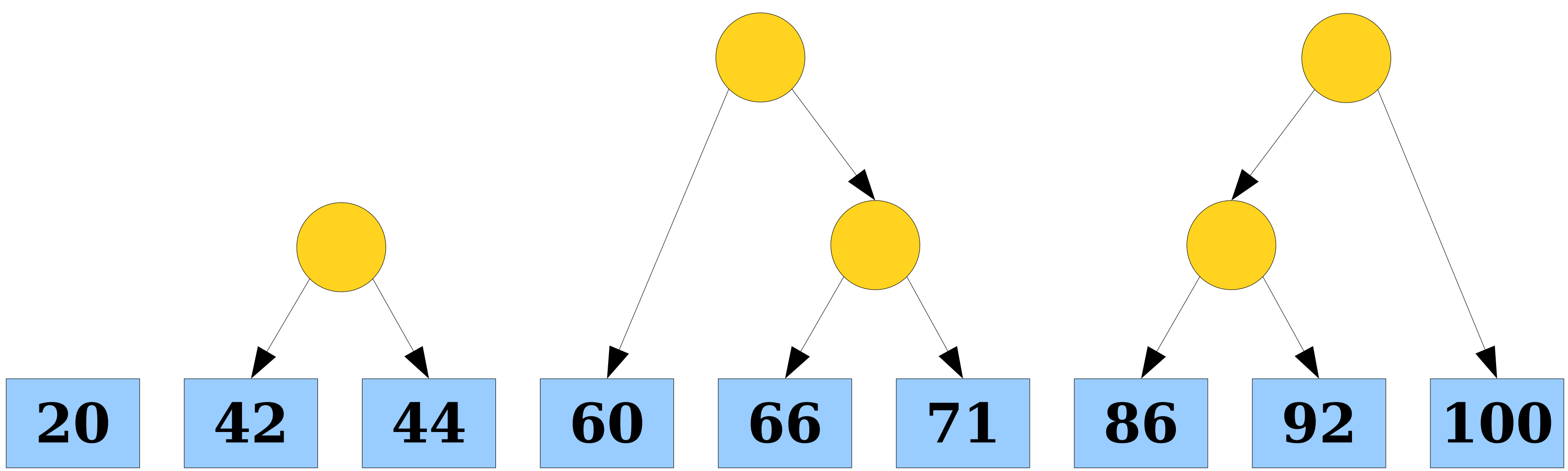
	42	44	60	66	71	86	92	100



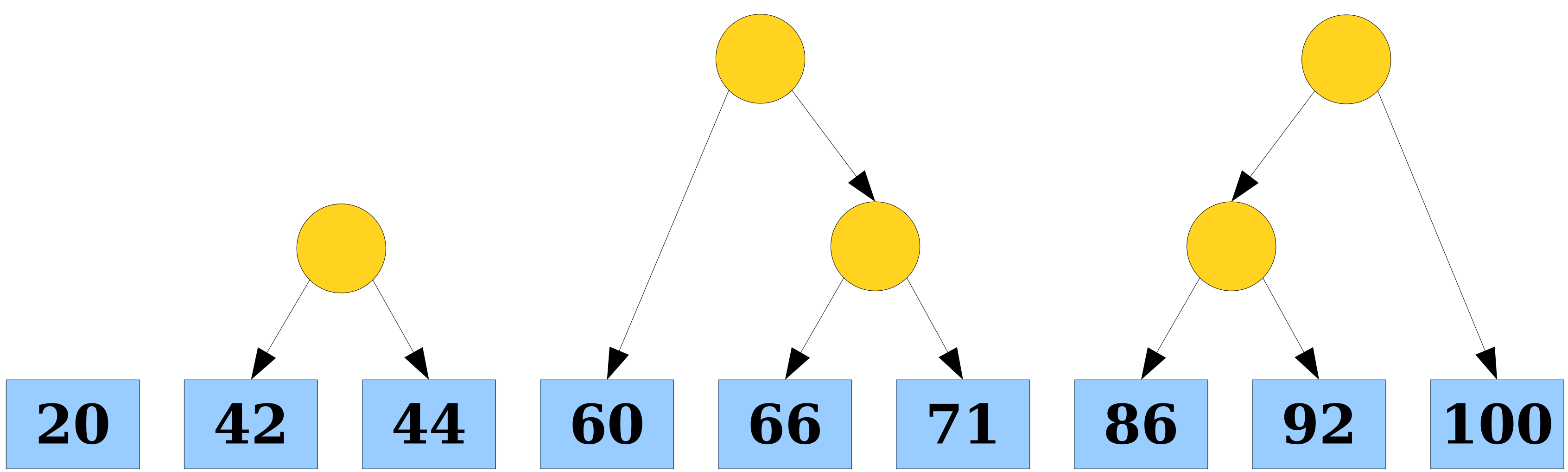
1D Hierarchical Clustering

20	**43**		**65.67**			**92.67**		
	42	**44**	**60**	**66**	**71**	**86**	**92**	**100**



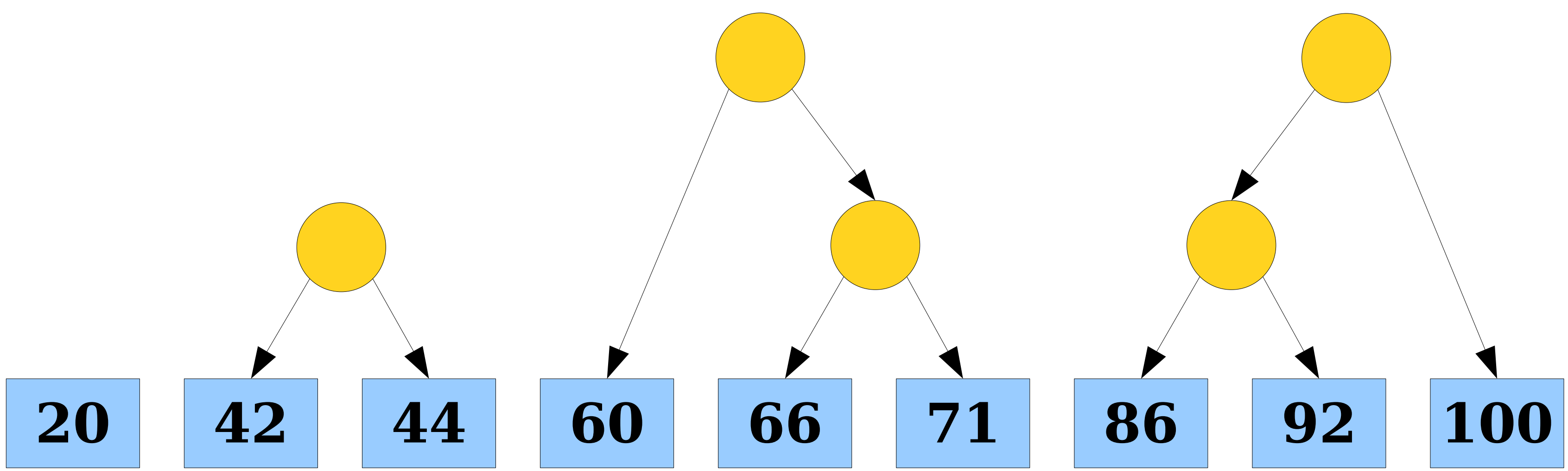
1D Hierarchical Clustering

20	43		65.67			92.67		
	42	44	60	66	71	86	92	100



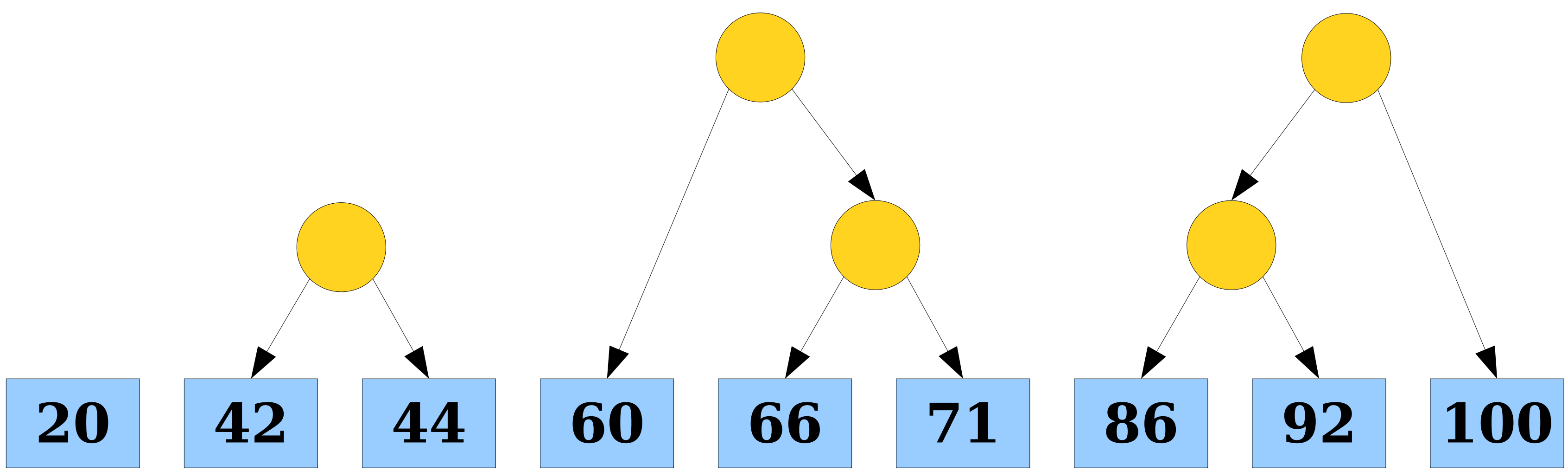
1D Hierarchical Clustering

20	**43**		**65.67**			**92.67**		
	42	**44**	**60**	**66**	**71**	**86**	**92**	**100**



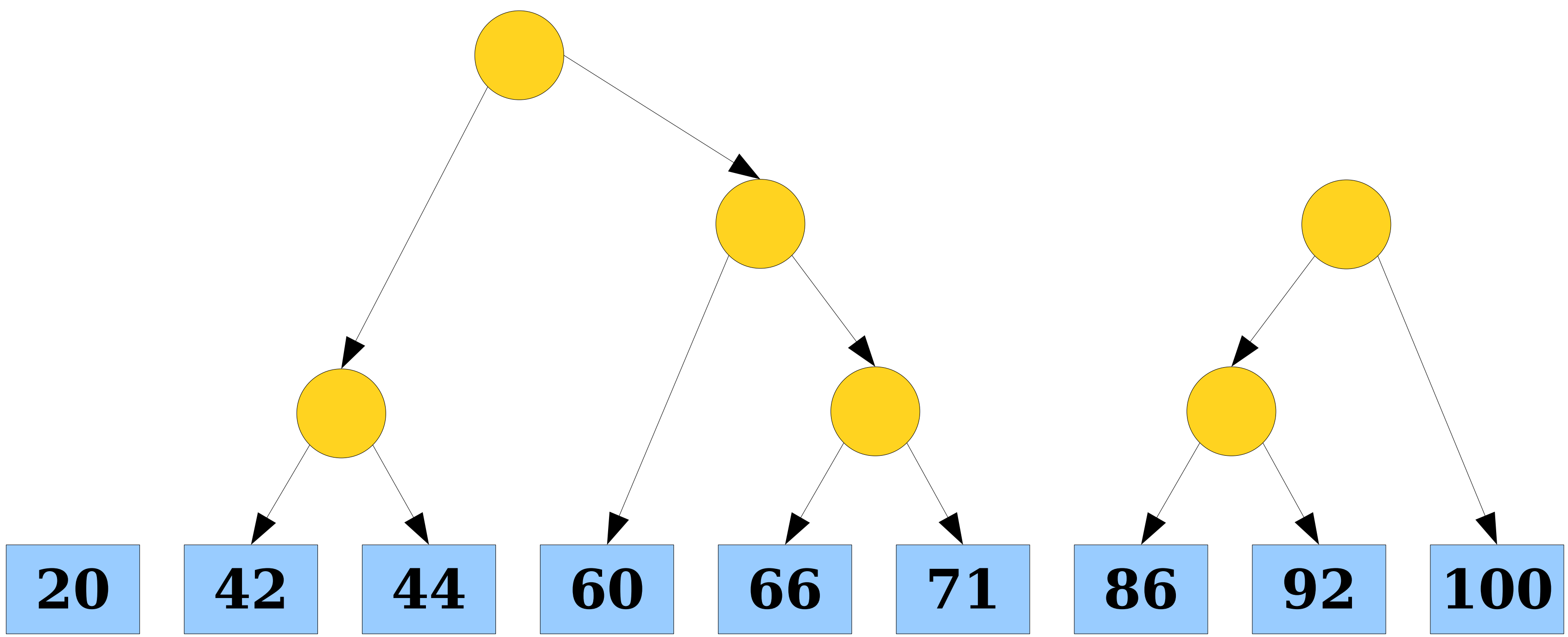
1D Hierarchical Clustering

20	**56.6**					**92.67**		
	42	**44**	**60**	**66**	**71**	**86**	**92**	**100**



1D Hierarchical Clustering

20	**56.6**					**92.67**		
	42	**44**	**60**	**66**	**71**	**86**	**92**	**100**



1D Hierarchical Clustering

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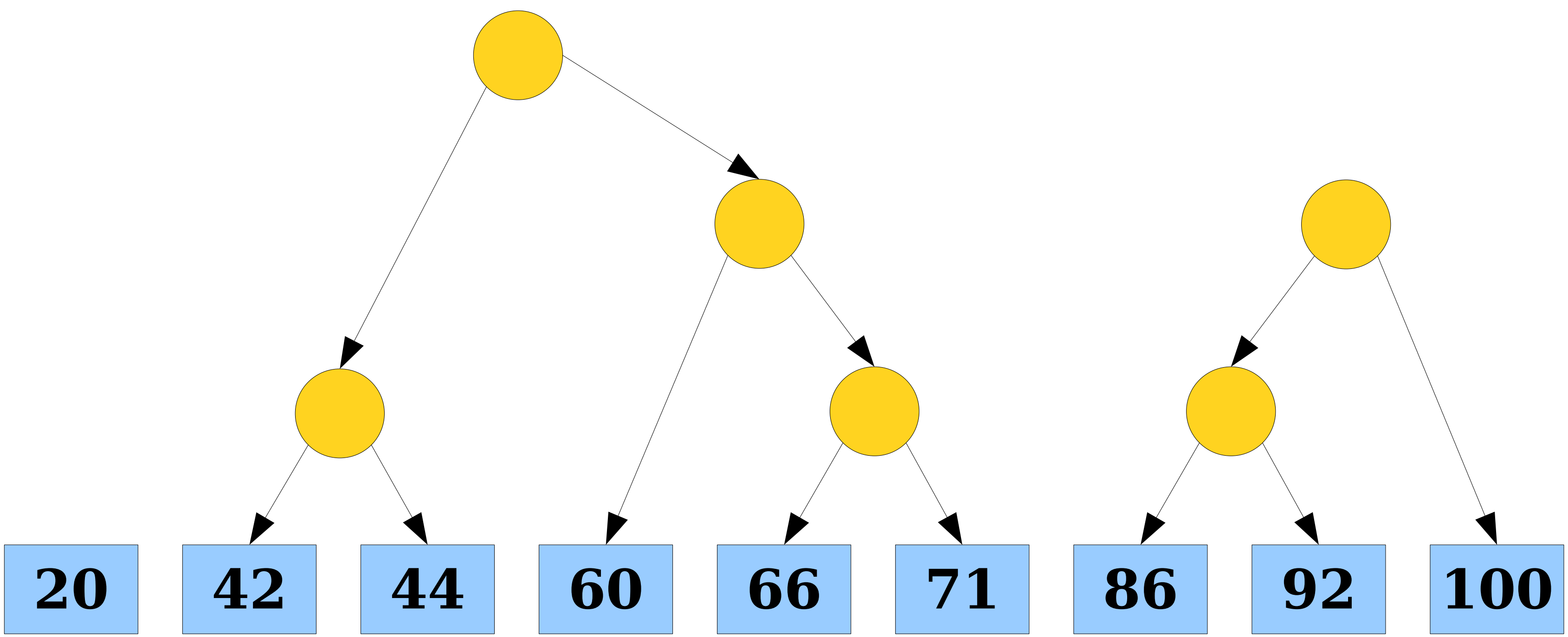

1D Hierarchical Clustering

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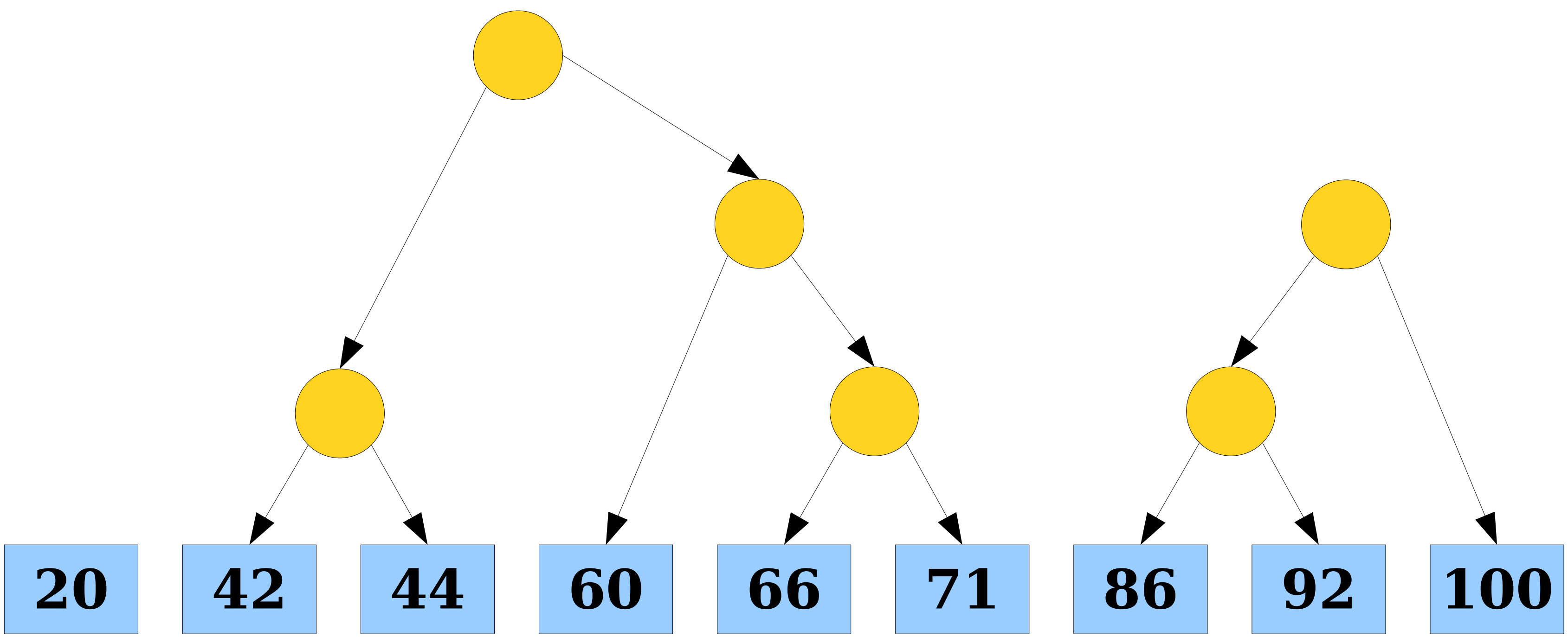
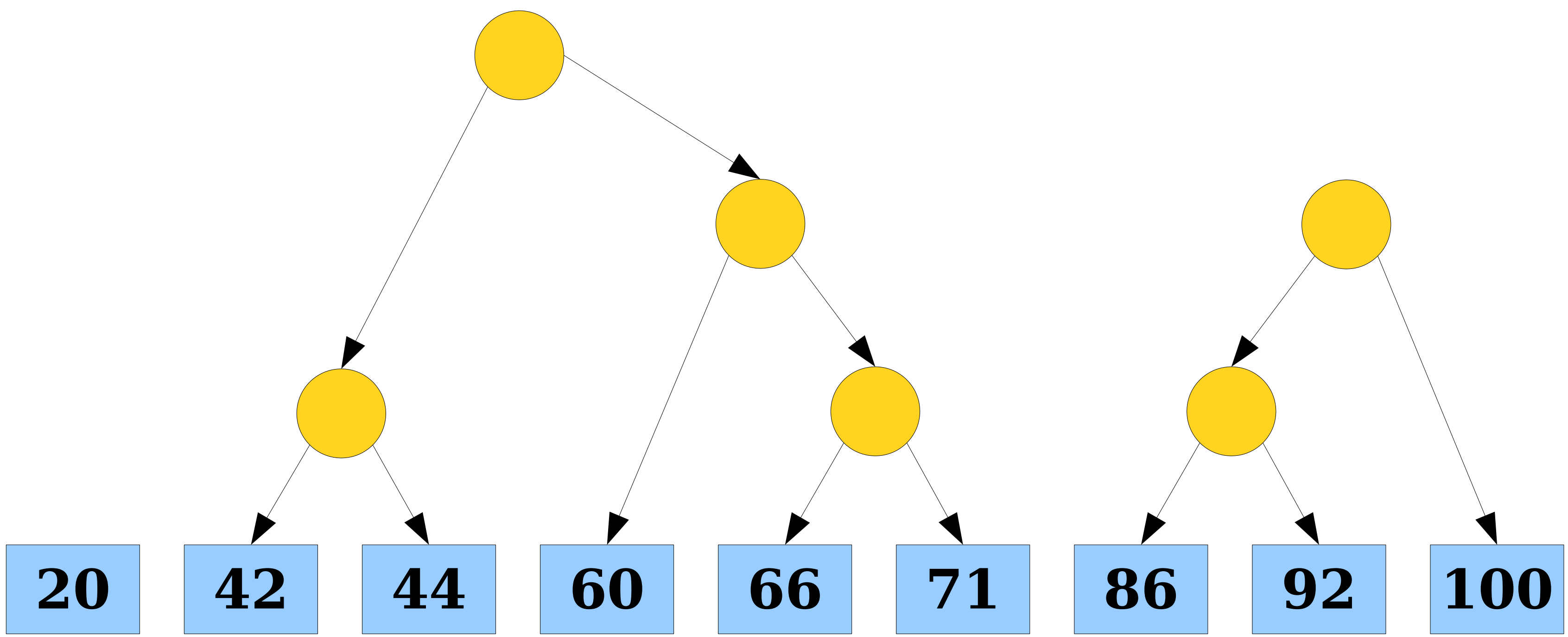


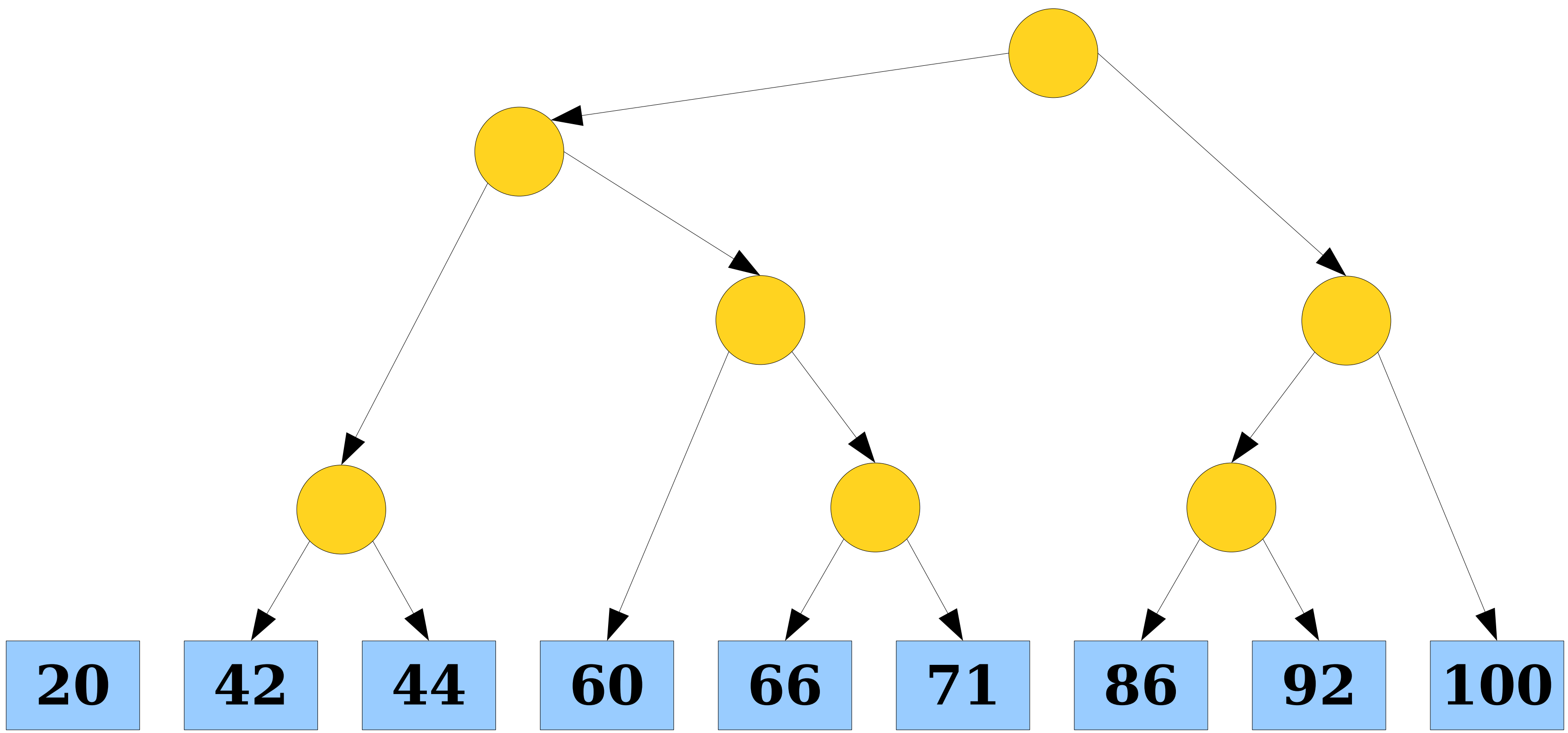
1D Hierarchical Clustering

20	70.13							
	42	44	60	66	71	86	92	100



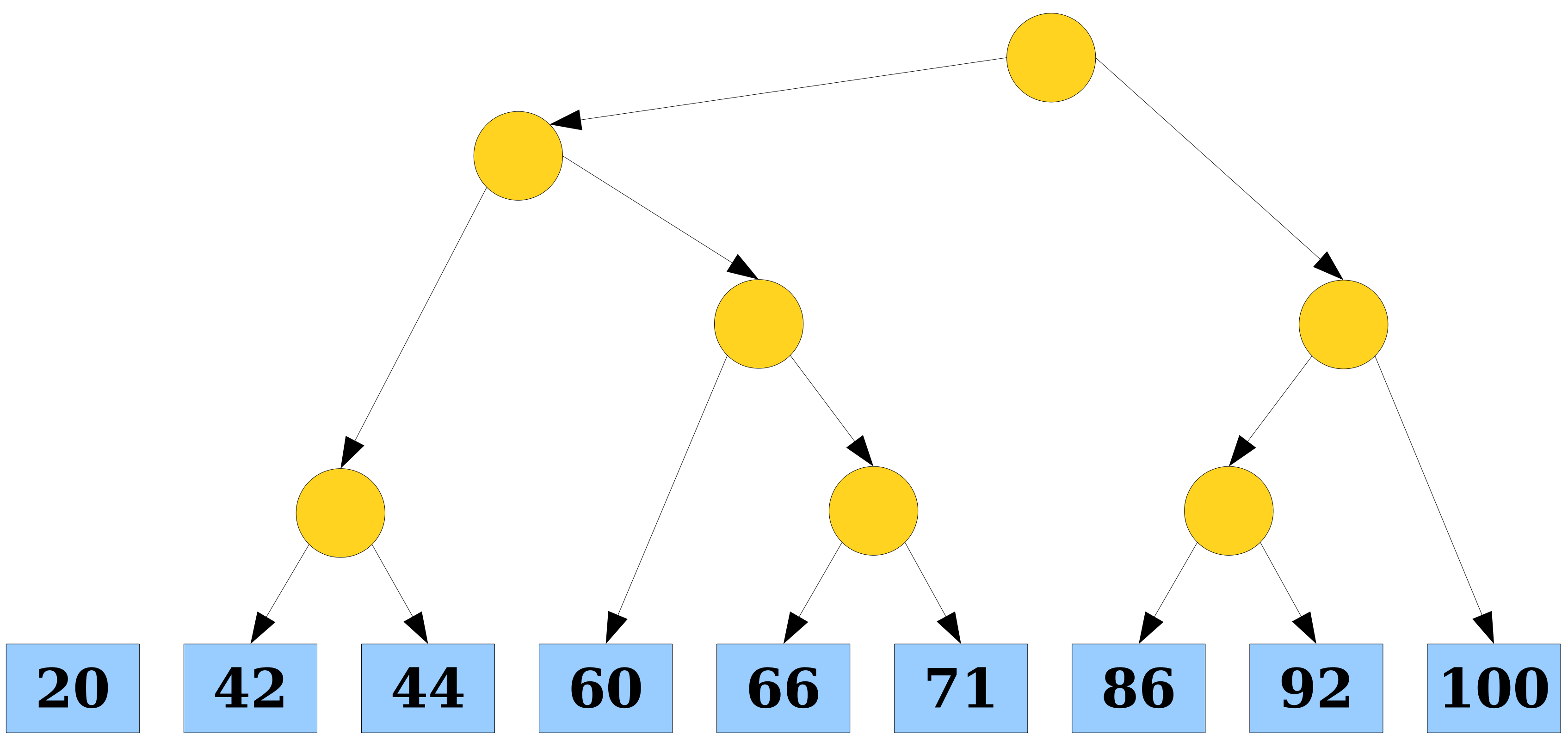
1D Hierarchical Clustering

20	**70.13**							
	42	**44**	**60**	**66**	**71**	**86**	**92**	**100**



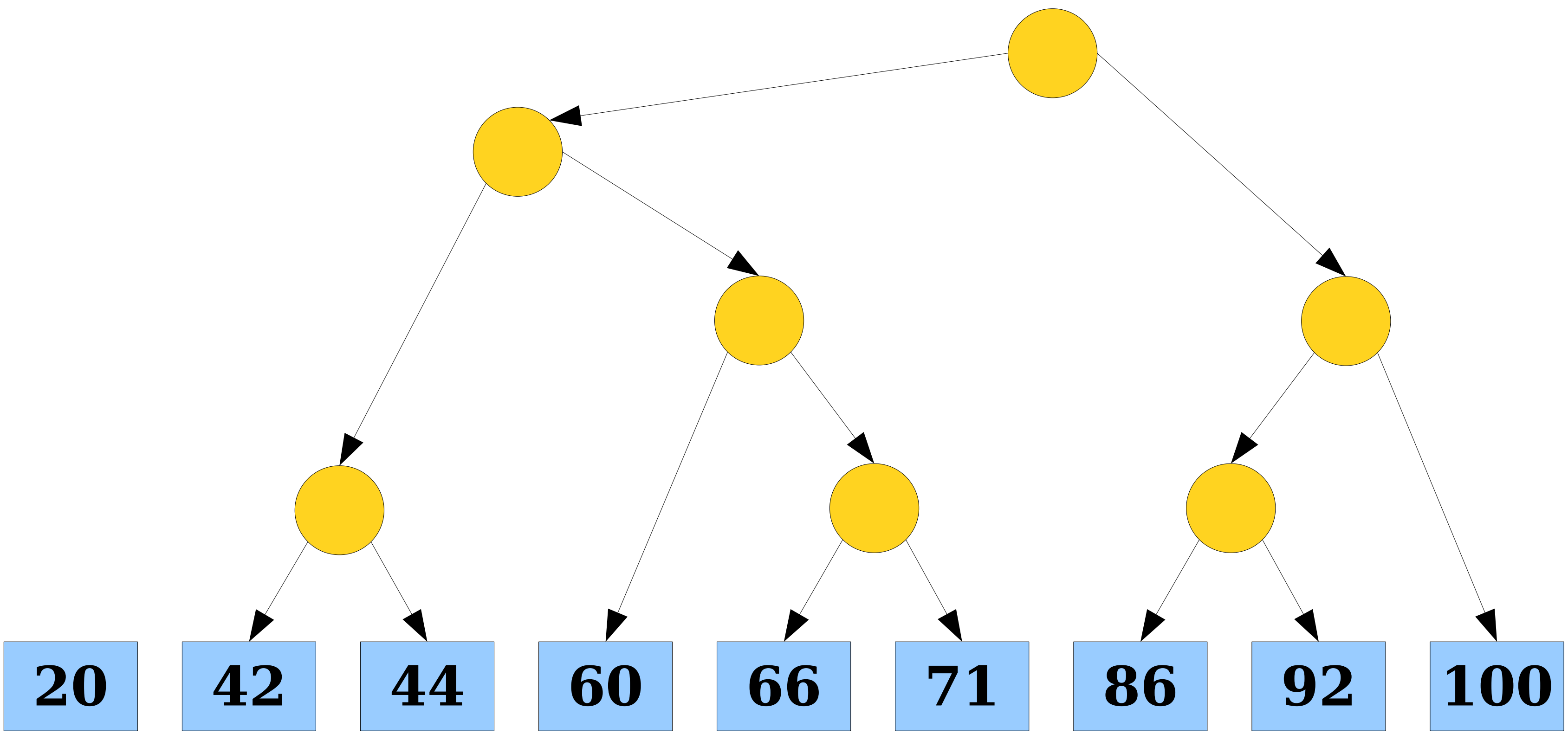
1D Hierarchical Clustering

20	70.13							
	42	44	60	66	71	86	92	100



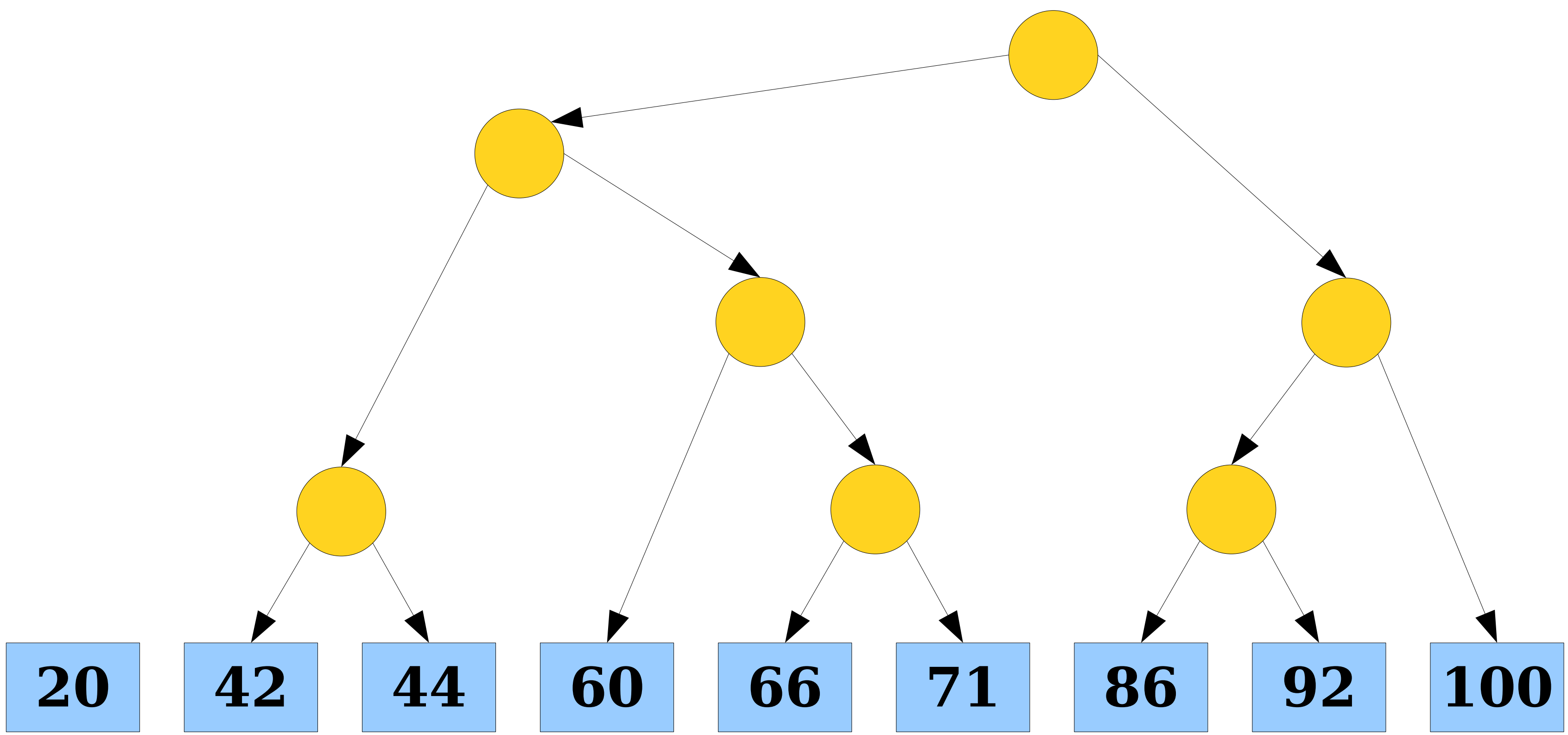
1D Hierarchical Clustering

20	**70.13**							
	42	**44**	**60**	**66**	**71**	**86**	**92**	**100**



1D Hierarchical Clustering

64.56								
20	42	44	60	66	71	86	92	100



1D Hierarchical Clustering

64.56								
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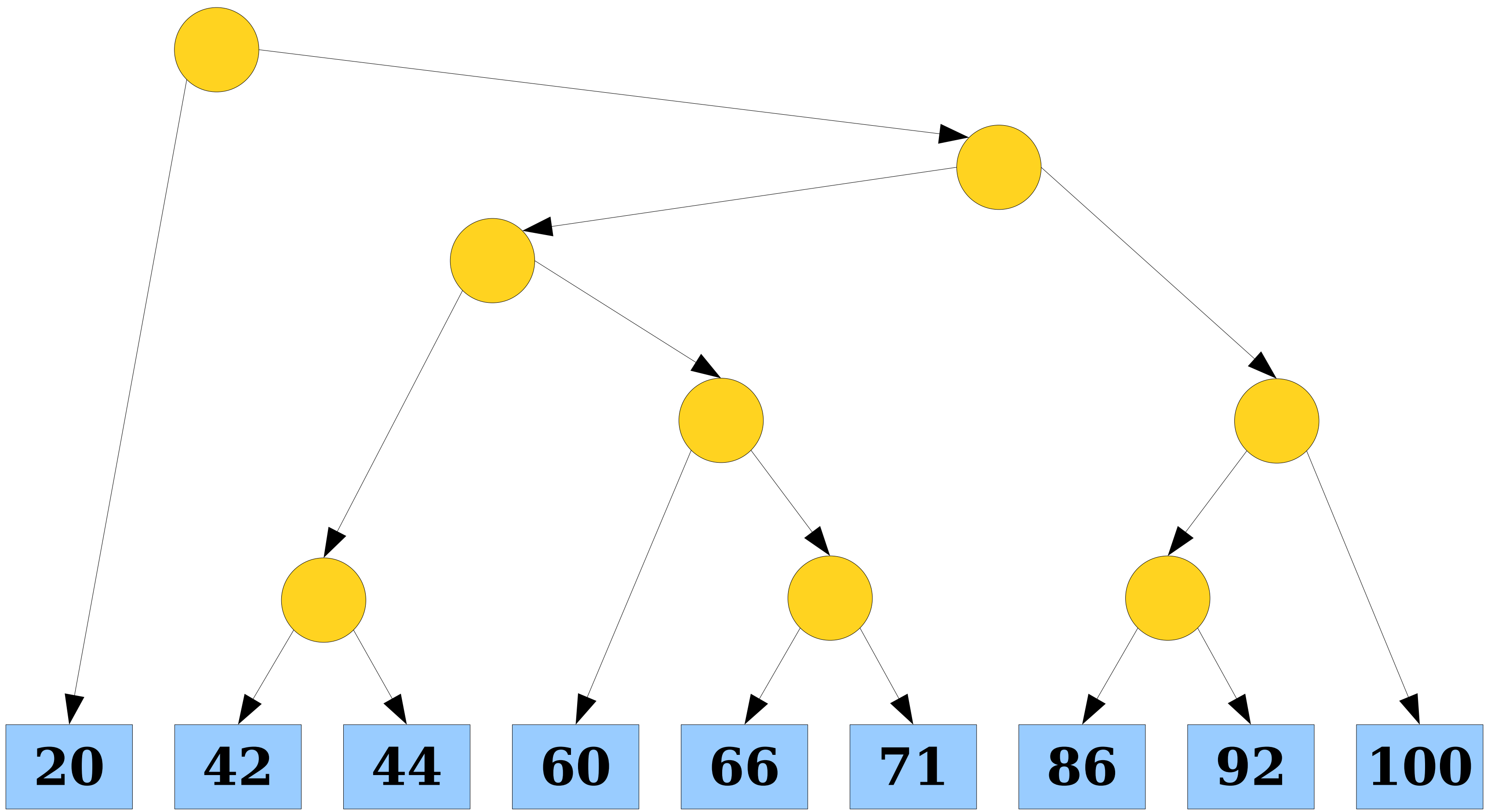
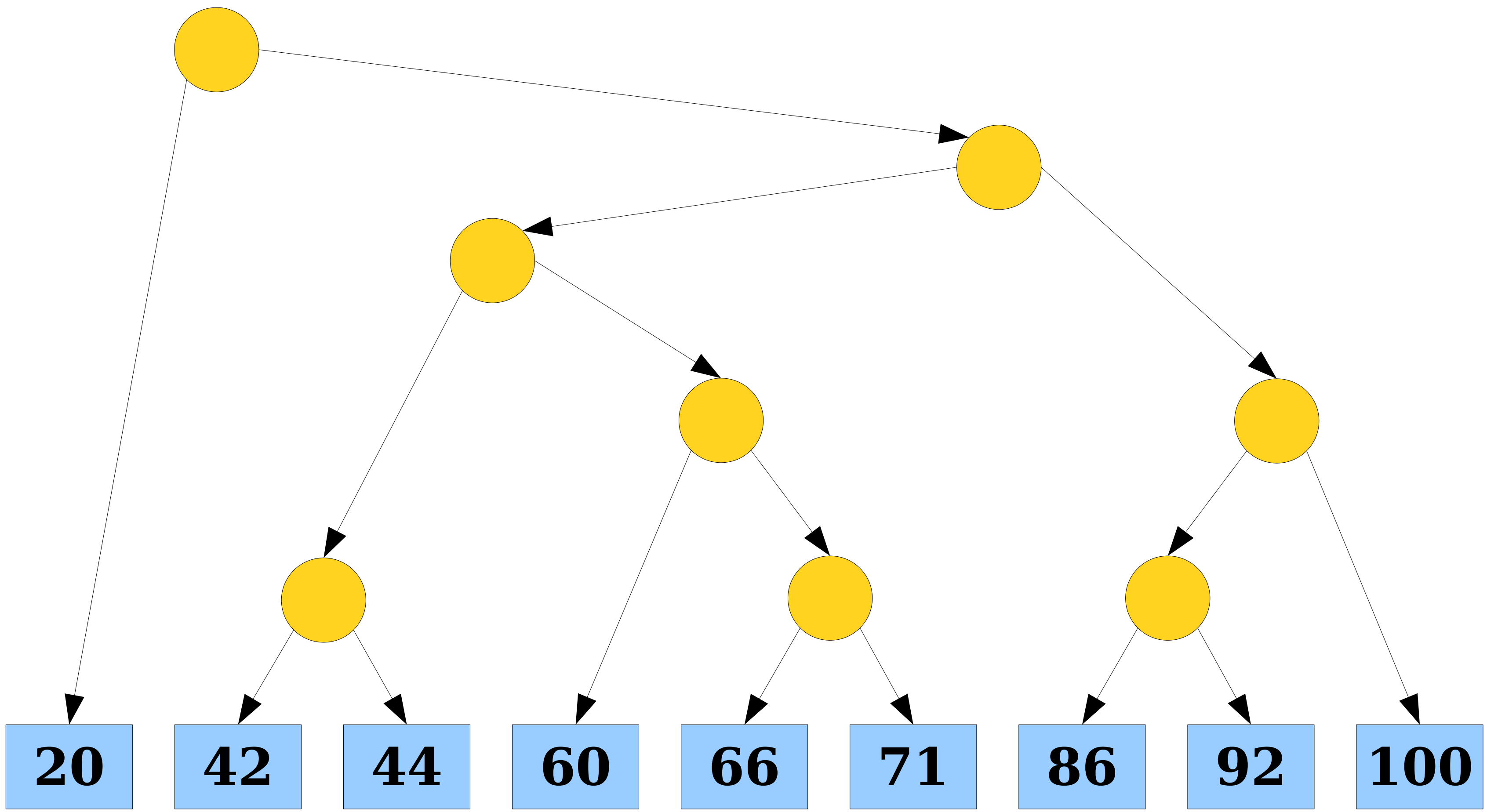
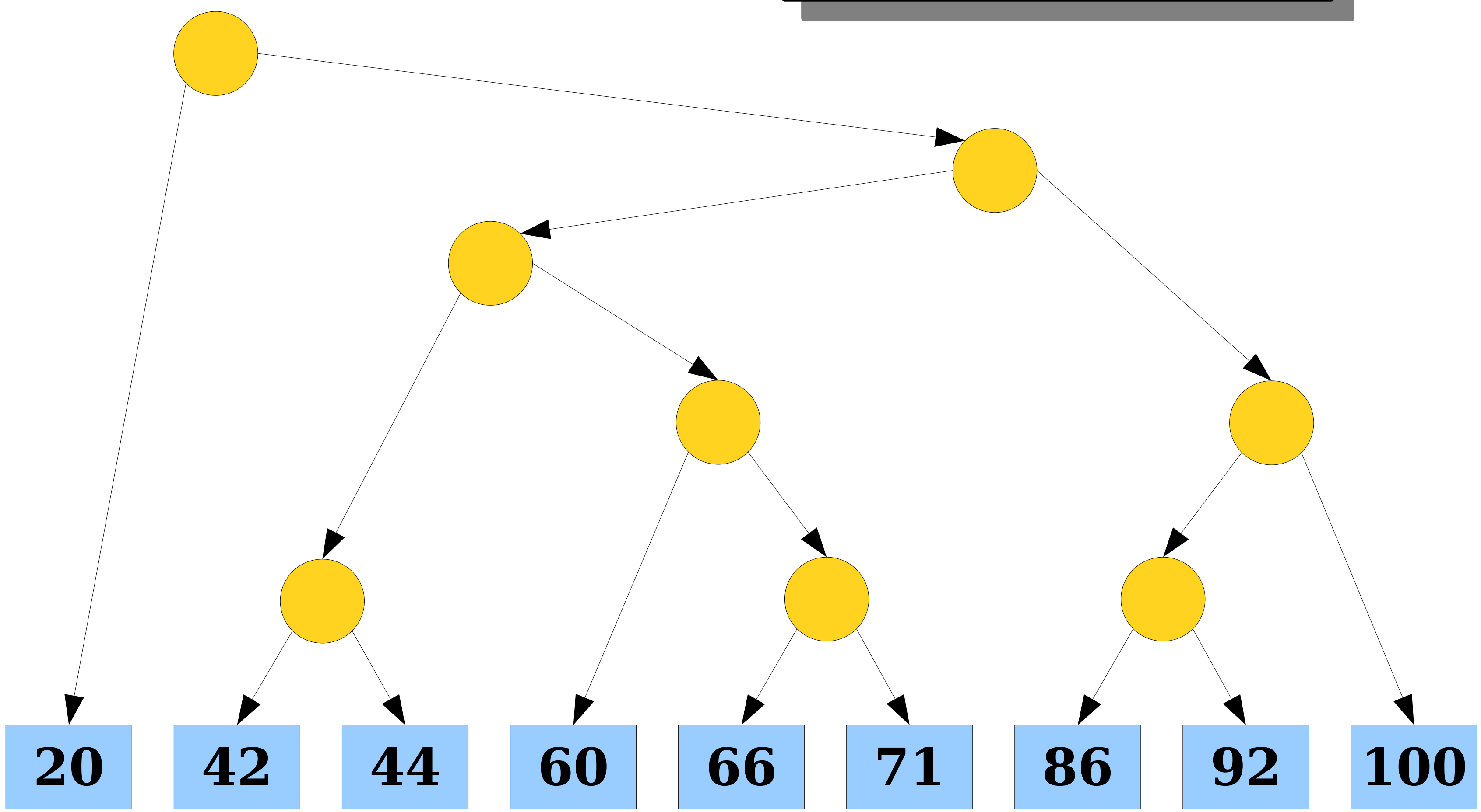


1D Hierarchical Clustering



1D Hierarchical Clustering

This tree is called a ***dendrogram***.



Analyzing the Runtime

- How efficient is this algorithm?
  - Number of rounds: $\Theta(n)$.
  - Work to find closest pair: $O(n)$.
  - Total runtime: $\boldsymbol{\Theta(n^2)}$.
- Can we do better?

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Can we do better?

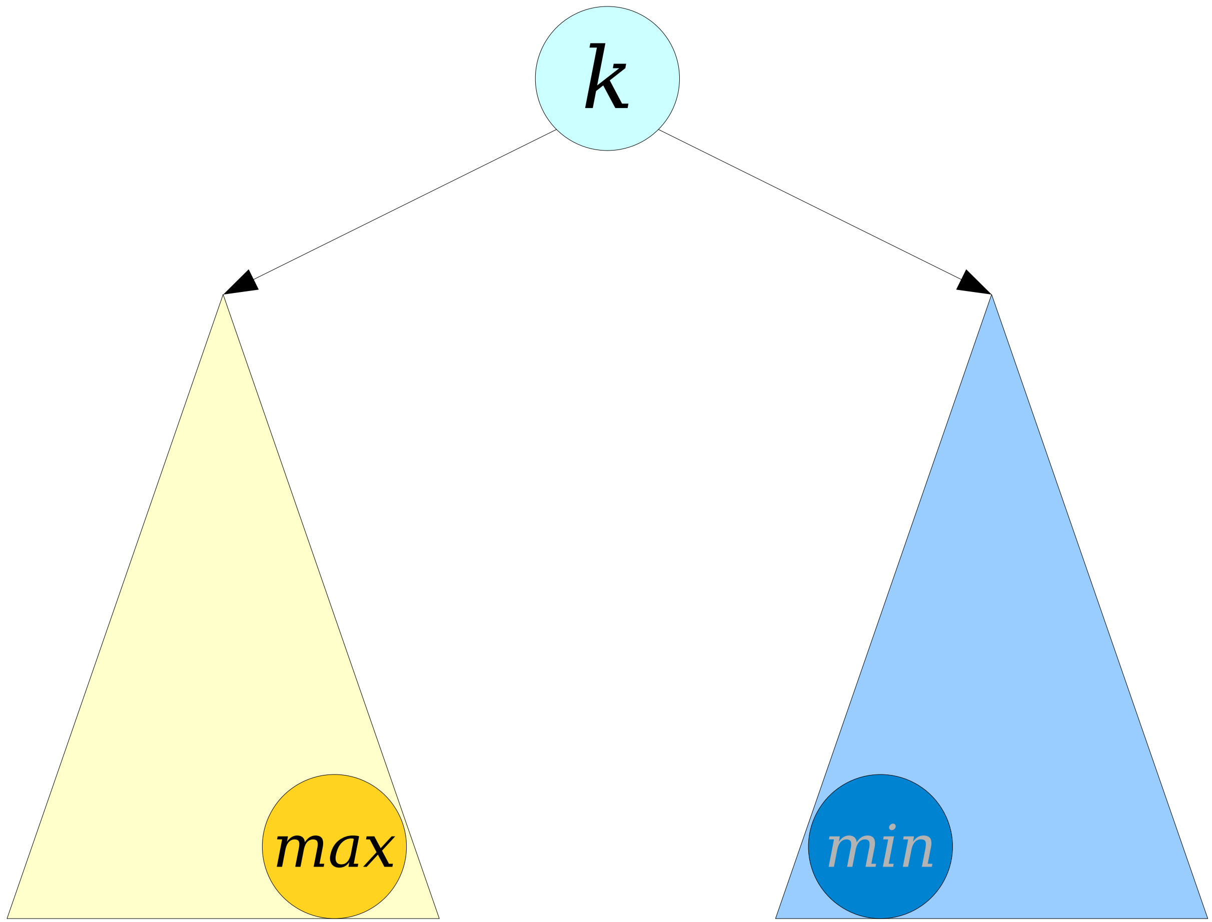
Dynamic 1D Closest Points

- The ***dynamic 1D closest points problem*** is the following:

  Maintain a set of real numbers undergoing insertion and deletion while efficiently supporting queries of the form "what is the closest pair of points?"

- Can we build a better data structure for this?

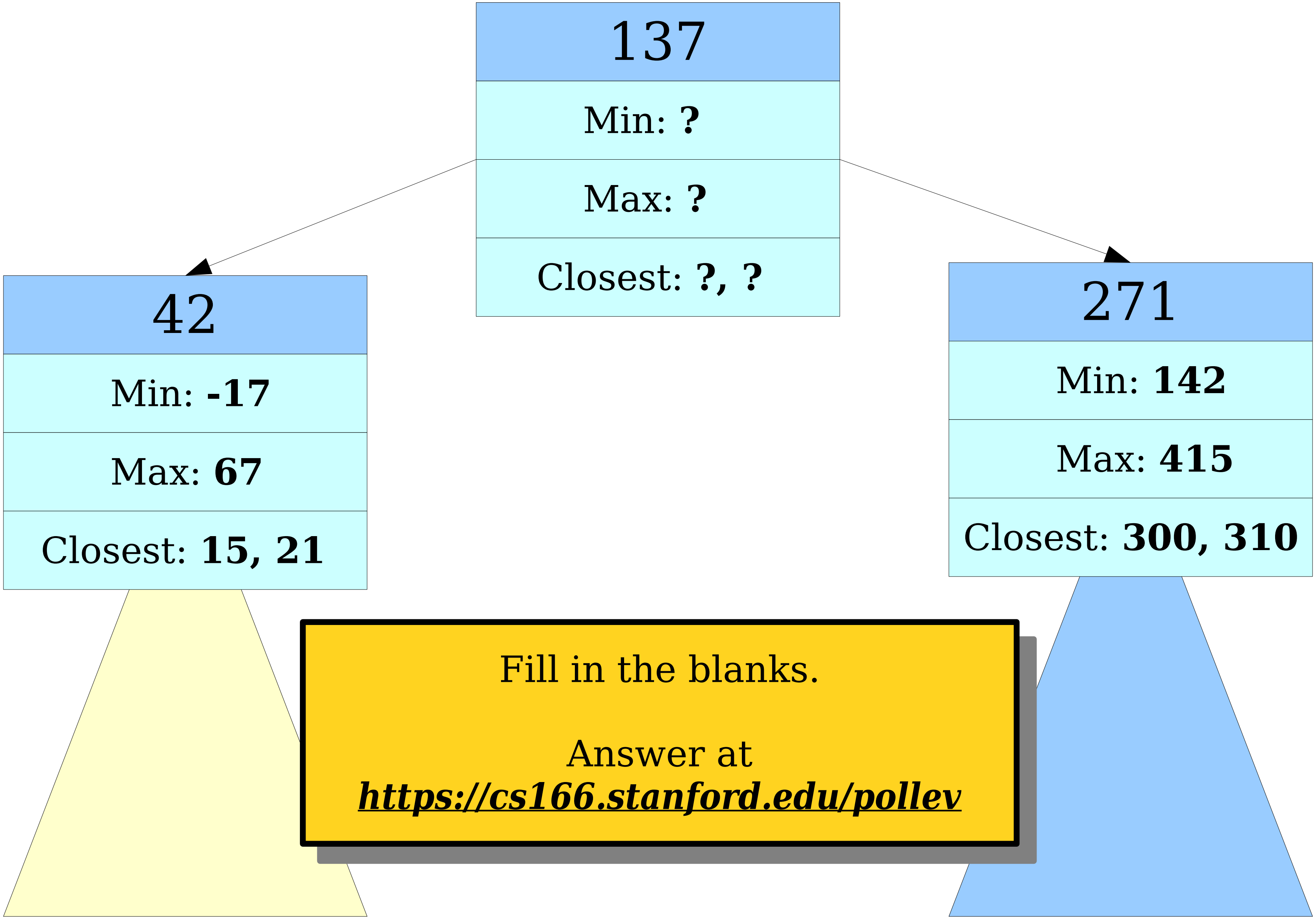
Dynamic 1D Closest Points



A Tree Augmentation

- Augment each node to store the following:
  - The maximum value in the tree.
  - The minimum value in the tree.
  - The closest pair of points in the tree.
- ***Claim:*** Each of these properties can be computed in time O(1) from the left and right subtrees.
- These properties can be augmented into a red/black tree so that insertions and deletions take time O(log *n*) and "what is the closest pair of points?" can be answered in time O(1).

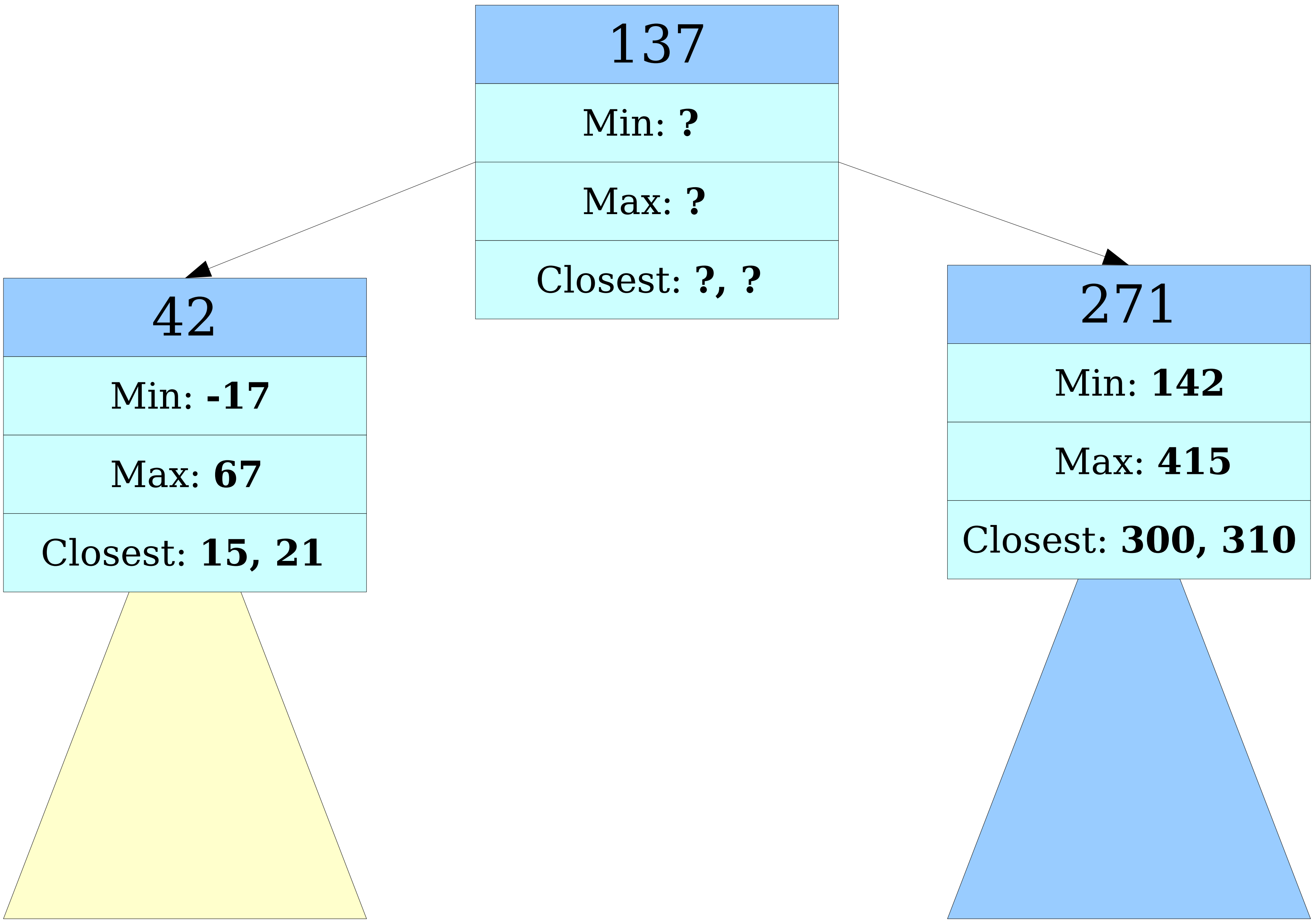
Dynamic 1D Closest Points



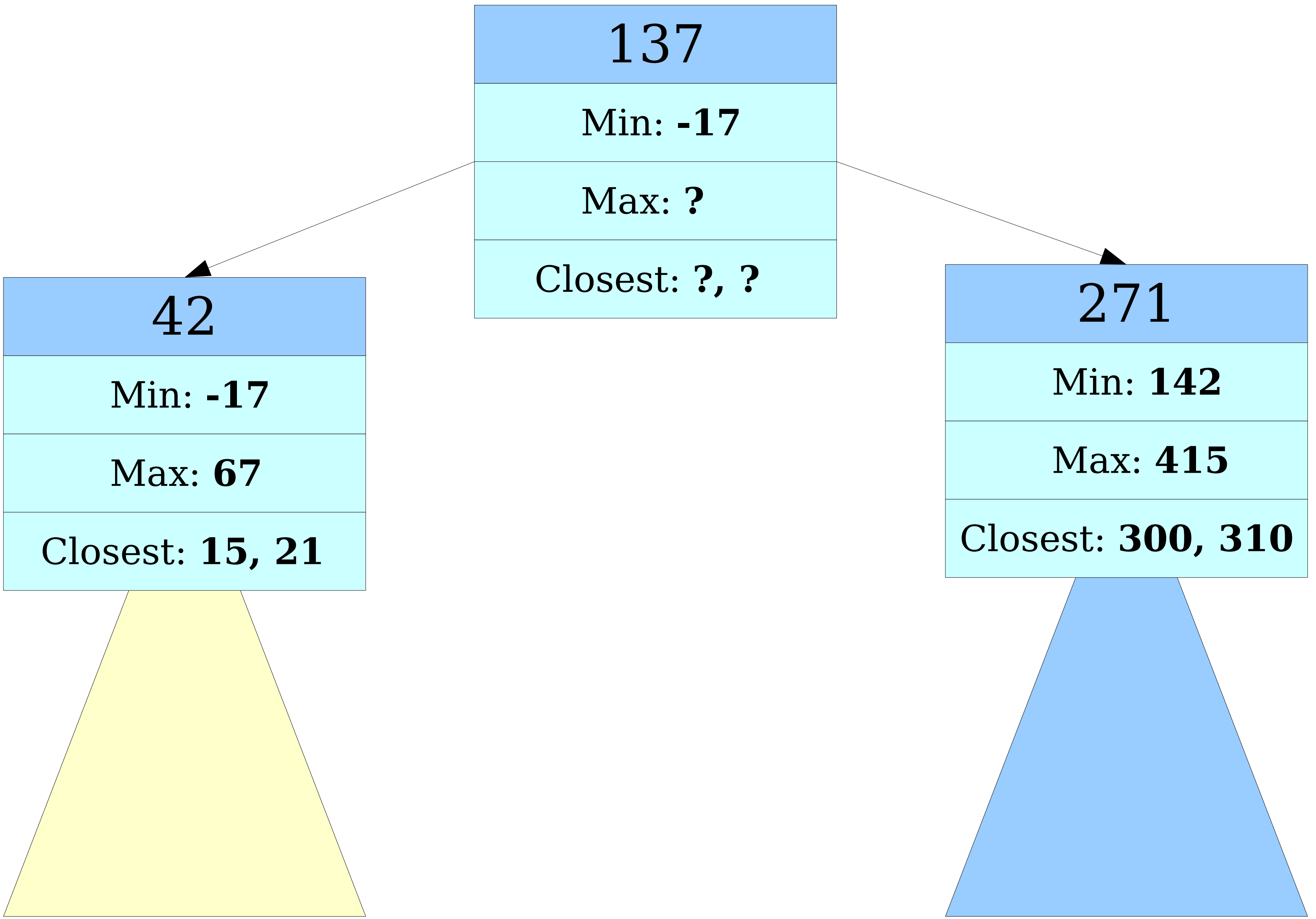
Fill in the blanks.

Answer at
https://cs166.stanford.edu/pollev

Dynamic 1D Closest Points



Dynamic 1D Closest Points



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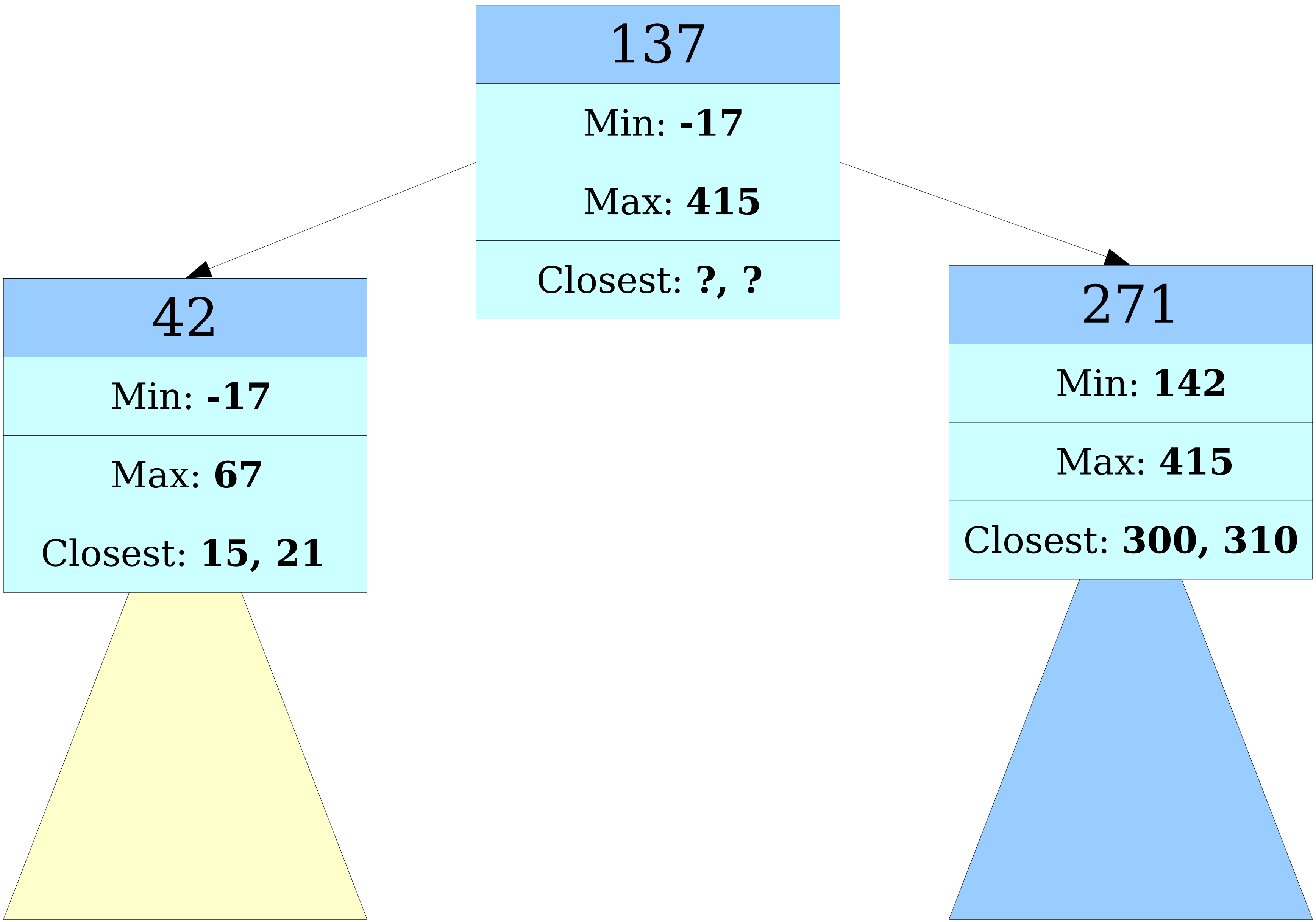
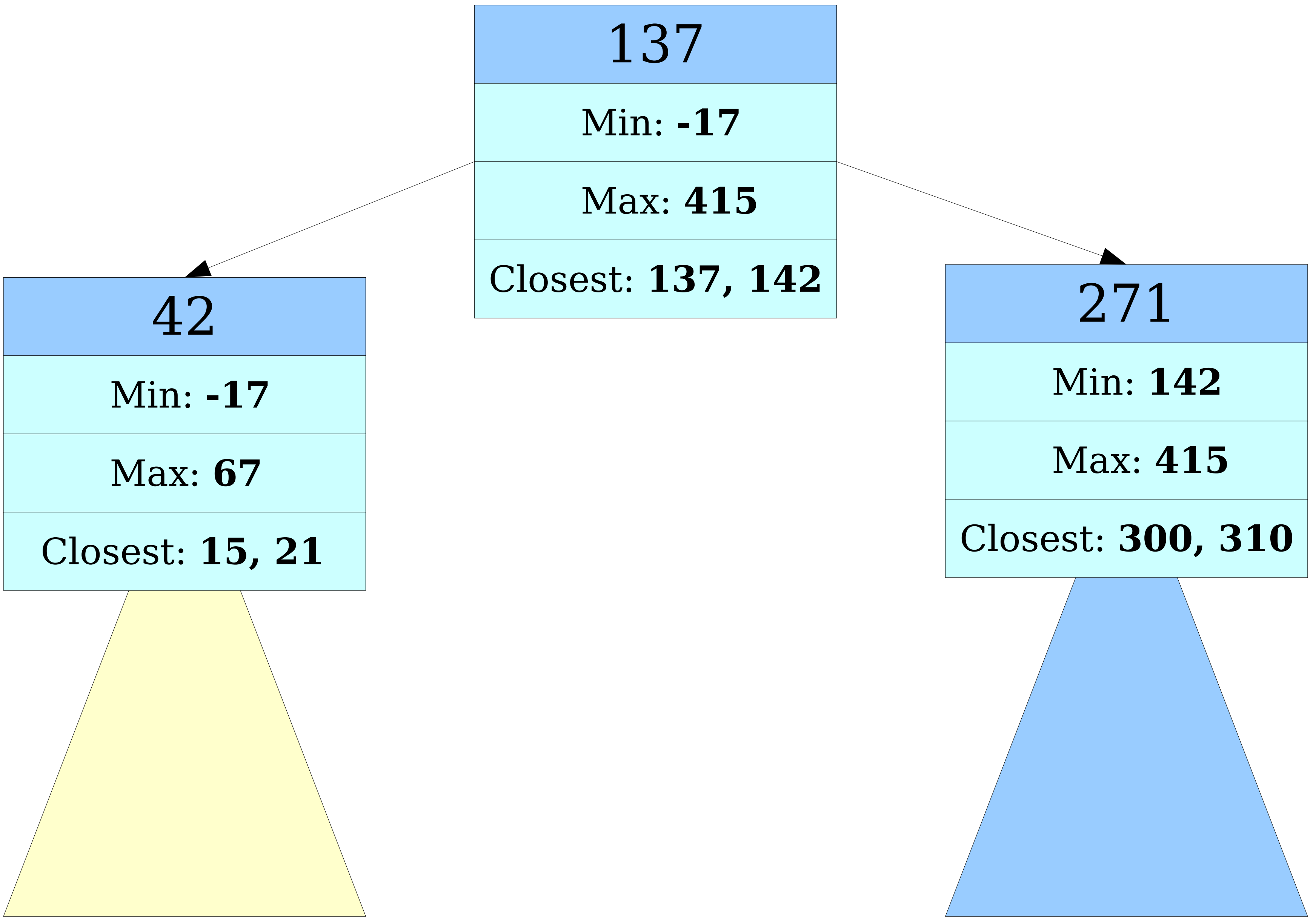


Dynamic 1D Closest Points



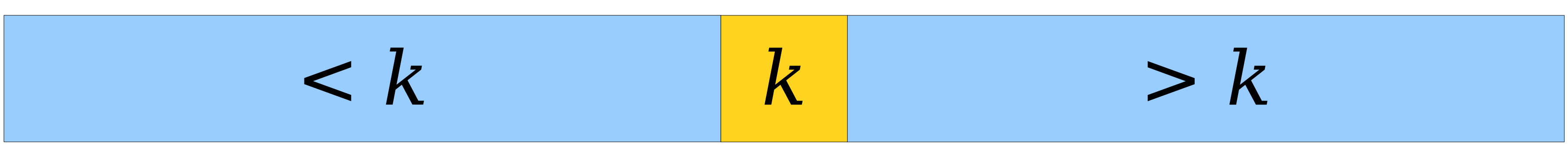
Some Other Questions

- How would you augment this tree so that you can efficiently (in time O(1)) compute the appropriate weighted averages?
- ***Trickier:*** Is this the fastest possible algorithm for this problem?
  - What if you're guaranteed that the keys are all integers in some nice range?

A Helpful Intuition

Divide-and-Conquer

- Initially, it can be tricky to come up with the right tree augmentations.
- ***Useful intuition:*** Imagine you're writing a divide-and-conquer algorithm over the elements and have O(1) time per “conquer” step.



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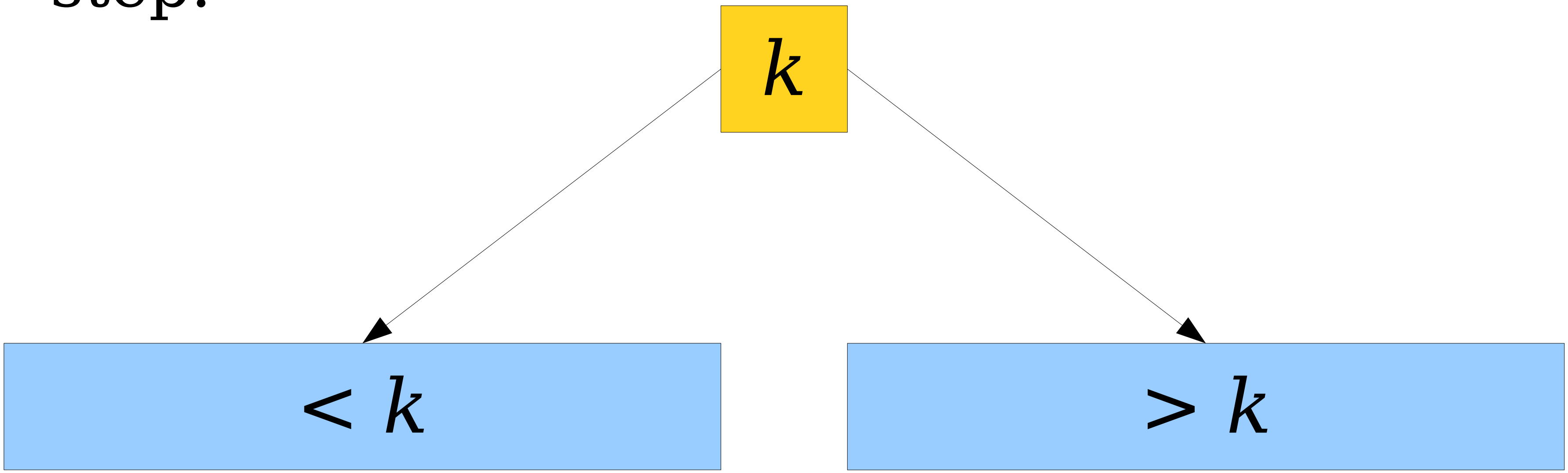

Next Time

- ***Randomized BSTs***
  - What do BSTs look like on random data?
- ***Markov on Moments***
  - A powerful generalization.